2

Income Transformations

Redistributions of income according to tax or transfer policies can be considered as variable transformations of the initial income. The transformation is usually assumed to be positive, monotone-increasing and continuous. Recently, Fellman (2009, 2011) has also discussed discontinuous transformations. If the transformation is considered as a tax or a transfer policy, the transformed variable is either the post-tax or the post-transfer income. A central problem in the literature has been the Lorenz dominance, defined above, between the initial and the transformed income (c.f. Fellman, 1976; Jakobsson, 1976; Kakwani, 1977) (see also Theorem 2.1.1 below). Under the assumption that the theorems should hold for all income distributions, the conditions are both necessary and sufficient (Jakobsson, 1976; Fellman, 2009).

2.1 Income Redistributions

Variable transformations are valuable when one studies the effect of tax and transfer policies on the income inequality. If the transformation should result in an increasing transformed variable with finite mean then discontinuities can only consist of finite positive jumps and the number of jumps has to be finite or countable. In this study we reconsider the effect of variable transformations on the redistribution of income (Fellman, 1976, Jakobsson, 1976, Kakwani, 1977 and Hemming & Keen, 1983). The continuity of the transformations can be implicitly included in the necessary and sufficient conditions. One main result is that continuity is a necessary condition if one pursues that the income inequality should remain or be reduced.

Consider the income X with the distribution function $F_X(x)$, the mean μ_X, and the Lorenz curve $L_X(p)$. We assume that X is defined for $x \geq 0$. If we assume that the density function $f_X(x)$ exists, we follow Section 1.1 and obtain the formulae

$$\mu_X = \int_0^\infty x\, f_X(x)dx \tag{2.1.1}$$

and

$$L_X(p) = \frac{1}{\mu_X}\int_0^{x_p} x\, f_X(x)dx. \tag{2.1.2}$$

We consider the transformation $Y = u(X)$, where $u(\cdot)$ is non-negative and monotone increasing. The transformation can be considered as a tax or a transfer policy and consequently, the transformed variable is the post-tax or post-transfer income, respectively.

For the transformed variable *Y* we obtain the distribution function

$$F_Y(y) = P(Y \le y) = P(u(X) \le y) = P\left(X \le u^{-1}(y)\right) = F_X\left(u^{-1}(y)\right) \tag{2.1.3}$$

Using this result we obtain the mean and the Lorenz curve for the variable *Y*.

$$\mu_Y = \int_0^1 u(x) f_X(x)dx \tag{2.1.4}$$

and

$$L_Y(p) = \frac{1}{\mu_Y}\int_0^{x_p} u(x) f_X(x)dx. \tag{2.1.5}$$

The fundamental theorem is:

Theorem 2.1.1. (Fellman, 1976, Jakobsson, 1976 and Kakwani, 1977). Let X be an arbitrary non-negative, random variable with the distribution $F_X(x)$, mean μ_X and Lorenz curve $L_X(p)$. Let $u(x)$ be non-negative, continuous and monotone-increasing and let $\mu_Y = E(u(X))$ exist. Then the Lorenz curve $L_Y(p)$ of $Y = u(X)$ exists and the following results hold:

(i) $L_Y(p) \geq L_X(p)$ if $\frac{u(x)}{x}$ is monotone decreasing.

(ii) $L_Y(p) = L_X(p)$ if $\frac{u(x)}{x}$ is constant and

(iii) $L_Y(p) \leq L_X(p)$ if $\frac{u(x)}{x}$ is monotone increasing.

According to this theorem we obtain in (i) a sufficient condition that the transformation $u(x)$ generates a new income distribution which Lorenz dominates the initial one. If we analyse the proof of the case (i) in Fellman (1976, Theorem 1) we observe that the difference $L_Y(p) - L_X(p)$ can be written

$$D(p) = L_Y(p) - L_X(p) = \int_0^{x_p} \left(\frac{u(x)}{\mu_Y} - \frac{x}{\mu_X} \right) f_X(x) dx \qquad (2.1.6)$$

where $x_p = F_X^{-1}(p)$. In any case, $D(0) = D(1) = 0$. In order to obtain Lorenz dominance the difference $D(p)$ must start from zero and then attain positive values and after that decrease back to zero and the integrand in (2.1.6) must start from positive (non-negative) values and then change its sign and become negative. Consequently, $\frac{u(x)}{x}$ has to be a decreasing function.

The condition is necessary if the rule should hold for all income distributions $F_X(x)$ (Jakobsson, 1976). Otherwise we can find a transformation $u(x)$ for which the quotient $\frac{u(x)}{x}$ is not monotone decreasing for all $x > 0$, and a distribution $F_X(x)$ such that the result in the proof holds, i.e. dominance is obtained. Assume that the quotient $\frac{u(x)}{x}$ is both increasing and decreasing.[3] Let a transformation $u(x)$ satisfy the initial conditions (non-negative, continuous and monotone increasing) and let $\frac{u(x)}{x}$ be increasing within some interval ($0 < a < x < b < \infty$). Now we present a distribution such that the transformed variable $Y = u(X)$ does not Lorenz dominate the initial variable *X*. Consider a distribution with a continuous density function,

$$f_X(x) = \begin{cases} 0 & 0 \le x < a \\ f_0(x) > 0 & a \le x \le b \\ 0 & x > b \end{cases} \tag{2.1.7}$$

For the pair ($f_X(x), u(x)$) the formula (2.1.6) can be written

$$D(p) = \int_a^{x_p} \frac{x}{\mu_Y}\left(\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X}\right) f_X(x)dx, \tag{2.1.8}$$

where $a \le x_p \le b$.

We observe that $D(0) = D(1) = 0$, that Theorem 1(iii) holds and that the transformation results in a new variable *Y* which is Lorenz dominated by the initial variable *X*. Hence, if we demand that the transformed variable $Y = u(X)$ shall Lorenz dominate *X* for all distributions $F_X(x)$, then the condition in Theorem 2.1.1 (i) is necessary (Jakobsson, 1976, Lambert, 2001; Chapter 8).

[3]If $\frac{u(x)}{x}$ is monotonously increasing for all $x > 0$ then the proposition (iii) holds and this case can be ignored.

Hemming and Keen (1983) gave a new condition for the Lorenz dominance. Their condition is, with our notations, that for a given distribution $F_X(x)$ the function $u(x)$ crosses the line $\frac{\mu_Y}{\mu_X}x$ once from above, that is that $\frac{u(x)}{x}$ crosses the level $\frac{\mu_Y}{\mu_X}$ once from above. We observe that if their condition holds then the integrand in (2.1.6) starts from positive values changes its sign once and ends up with negative values and their condition is equivalent with our condition. For the example considered above, the Hemming-Keen condition is not satisfied. The integrand is zero for $x < a$ and for $x > b$. For $a \le x \le b$ the ratio $\frac{u(x)}{x}$ is increasing and if it crosses $\frac{\mu_Y}{\mu_x}$ it cannot do it from above. Consequently, if $\frac{u(x)}{x}$ is not monotone decreasing then there are distributions for which the Hemming-Keen condition does not hold.

On the other hand if we assume that $\frac{u(x)}{x}$ is monotone decreasing then $\frac{u(x)}{x}$ satisfies the condition "crossing once from above for every distribution $F_X(x)$". Hence, our condition and Hemming-Keen condition are also equivalent as necessary conditions.

In a similar way we can prove that if the other results in Theorem 2.1.1 should hold for every income distribution the conditions in (ii) and in (iii) are also necessary.

The results obtained, indicate that if $\frac{u(x)}{x}$ is continuous and monotone increasing even in a short interval, then there are income distributions such that the transformation $u(x)$ cannot result in Lorenz dominance. What can be said if $u(x)$ is discontinuous? Assume that $u(x)$ is still positive and monotone increasing. Assume furthermore, that $E(u(X)) = \mu_Y$ exists for every stochastic variable X with finite mean μ_X. Above we stressed that the discontinuities of $u(x)$ can only consist of finite positive jumps and the number of jumps can be assumed to be finite or countable. Assume that elsewhere $u(x)$ satisfies all the other conditions including the condition in Theorem 2.1.1(i). We will prove that if $u(x)$ is discontinuous there exists a distribution $F_X(x)$ such that the transformation $Y = u(X)$ does not Lorenz dominate the initial variable *X*. Again we follow the arguments given by Jakobsson (1976). However, the discontinuity demands a more detailed reasoning.

Let $a > 0$ be a discontinuity point, such that $lim_{x \to a-} u(x) = u_0$ and $lim_{x \to a+} u(x) = u_0 + d$, where the jump $d > 0$. (The notation $lim_{x \to a-} u(x)$ indicates limit from the left and $lim_{x \to a+} u(x)$ limit from the right.) We do not assume anything about how $u(x)$ is defined in the point a. The following analyses are based on Fellman (2009). Choose $h > 0$ so small that the point a is the only discontinuity point within the interval $(a-h, a+h)$. (Later we may reduce the interval even more). Let t and z be arbitrary values satisfying the inequalities

$$a - h < t \le a \le z < a + h.$$

If $u(x)$ is monotone increasing we have $u(t) \le u_0 < u_0 + d \le u(z)$ and

$$lim_{t\to a-}\left(\frac{u(t)}{t}\right)=\frac{u_0}{a}<\frac{u_0+d}{a}=lim_{z\to a+}\left(\frac{u(z)}{z}\right).$$

Hence, the quotient $\frac{u(x)}{x}$ cannot be monotone decreasing within the interval $(a-h,a+h)$. Consider a variable *X*, having the symmetric density function

$$f_X(x)=\begin{cases} 0 & x<a-h \\ \frac{1}{h}\left(1-\frac{1}{h}|a-x|\right) & a-h\le x\le a+h \\ 0 & x>a-h \end{cases}. \qquad (2.1.9)$$

The mean $E(X)=\mu_X=a$. For the transformed variable $Y=u(X)$ the mean is

$$\begin{aligned}\mu_Y=E(Y)&=\int_{a-h}^{a}u(x)f_X(x)dx+\int_{a}^{a+h}u(x)f_X(x)dx\\ &=u(\alpha_1)\int_{a-h}^{a}f_X(x)dx+u(\alpha_2)\int_{a}^{a+h}f_X(x)dx \qquad , \qquad (2.1.10)\\ &=\frac{1}{2}(u(\alpha_1)+u(\alpha_2))\end{aligned}$$

where $a-h<\alpha_1<a$ and $a<\alpha_2<a+h$.

If $h\to 0$ then $\mu_Y\to u_0+\frac{1}{2}d$. Assume furthermore, that we have chosen *h* so small that $\mu_Y>u_0+\frac{1}{4}d$. Consider now

$$D(p)=L_Y(p)-L_X(p)=\int_{a-h}^{x_p}\frac{x}{\mu_Y}\left(\frac{u(x)}{x}-\frac{\mu_Y}{\mu_X}\right)f_X(x)dx, \qquad (2.1.11)$$

where $F_X(x_p) = p$. In order to obtain Lorenz dominance the integrand must start from positive (non-negative) values and then change its sign and become negative in such a manner that the difference $D(p)$ starts from zero and then attains positive values and after that it decreases back to zero. Within the interval $(a-h, a+h)$ the sign of the integrand depends on the factor $\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X}$, which starts from the value

$$\frac{u(a-h)}{a-h} - \frac{\mu_Y}{a} \le \frac{u_0}{a-h} - \frac{u_0 + \frac{1}{4}d}{a} \le \frac{-\frac{1}{4}ad + h(u_0 + \frac{1}{4}d)}{a(a-h)}.$$

If we assume that h satisfies the earlier conditions and in addition, the condition $h < \frac{ad}{4u_0 + d}$, the parenthesis in (2.1.11) starts from negative values and consequently, the whole integrand is negative and $D(p)$ starts from negative values. For the corresponding income distribution the transformed variable Y does not Lorenz dominate the initial variable X. Hence, the continuity of $u(x)$ is a necessary condition if we demand that the transformed variable should Lorenz dominate the initial variable for every distribution. From this it follows that if the condition in Theorem 2.1.1(i) has to be necessary it implies continuity and hence, an explicit statement of continuity can be dropped. If we study the condition in (ii) we observe that $u(x) = kx$ and consequently, $u(x)$ has to be continuous.

However, in the case (iii) the discontinuity does not jeopardize the monotone increasing property of the quotient $\frac{u(x)}{x}$ and the result in Theorem 2.1.1 (iii) holds even if the function is discontinuous. Therefore, also in this case we can drop the explicit continuity assumption.

Summing up, for arbitrary distributions, $F_X(x)$, the conditions (i), (ii), and (iii) in Theorem 2.1.1 are both necessary and sufficient for the dominance relations and an additional assumption about the continuity of the transformation $u(x)$ can be dropped. We obtain the more general theorem (Jakobsson, 1976; Fellman, 2009).

Theorem 2.1.2. Let X be an arbitrary non-negative, random variable with the distribution $F_X(x)$, mean μ_X and the Lorenz curve $L_X(p)$, let $u(x)$ be a non-negative, monotone increasing function and let $Y = u(X)$ and $E(Y) = \mu_Y$ exist. Then the Lorenz curve $L_Y(p)$ of Y exists and the following results hold:

(i) $L_Y(p) \geq L_X(p)$ if and only if $\dfrac{u(x)}{x}$ is monotone-decreasing.

(ii) $L_Y(p) = L_X(p)$ if and only if $\dfrac{u(x)}{x}$ is constant.

(iii) $L_Y(p) \leq L_X(p)$ if and only if $\dfrac{u(x)}{x}$ is monotone-increasing.

Remark. From the discussion above it follows that only in the case (iii) the transformation $u(x)$ can be discontinuous.

Now, we analyse the effect of a finite step in $u(x)$ on the Lorenz curve. We use the notations presented above.

Let $t \leq a \leq z$, $r = F_X(t)$, $q = F_X(a)$ and $s = F_X(z)$.

Consider the difference

$$\Delta L_Y = L_Y(F_X(z)) - L_Y(F_X(t)) = \frac{1}{\mu_Y}\int_t^z u(x) f_X(x)\,dx$$

$$= \frac{1}{\mu_Y}\int_t^a u(x) f_X(x)dx + \frac{1}{\mu_Y}\int_a^z u(x) f_X(x)dx = \frac{u(\alpha_1)}{\mu_Y}(q-r) + \frac{u(\beta_1)}{\mu_Y}(s-q)$$

where $t \le \alpha_1 \le a$ and $a \le \beta_1 \le z$.

When $t \to a-$ and $z \to a+$, then $q - r \to 0$, $s - q \to 0$ and $\Delta L_Y \to 0$. Hence, although the transformation $u(x)$ is discontinuous in the point a, the Lorenz curve is continuous. However, it is not differentiable. For every $t < a$ we obtain

$$\Delta L_Y = L_Y(q) - L_Y(r) = \frac{1}{\mu_Y}\int_t^a u(x) f_X(x)dp = \frac{u(\eta)}{\mu_Y}(q-r)$$

where $t < \eta < a$. We obtain $\frac{\Delta L_Y}{q-r} = \frac{u(\eta)}{\mu_Y}$. When $q - r \to 0+$ then $\eta \to a-$ and $\frac{\Delta L_Y}{q-r} \to \frac{u_0}{\mu_Y}$. Hence, $L_Y(p)$ has the left derivative $\left(\frac{dL_Y(p)}{dp}\right)_{p=q-} = \frac{u_0}{\mu_Y}$.

For every $z > a$ we obtain

$$\Delta L_Y = L_Y(s) - L_Y(q) = \frac{1}{\mu_Y}\int_q^s u(x) f_X(x)dp = \frac{u(\varsigma)}{\mu_X}(s-q),$$

where $a < \varsigma < z$. We obtain $\frac{\Delta L_Y}{s-q} = \frac{u(\varsigma)}{\mu_Y}$. When $s - q \to 0+$ then $\eta \to a+$ and $\frac{\Delta L_Y}{s-q} \to \frac{u_0 + d}{\mu_Y}$. Hence, $L_Y(p)$ has the right derivative

$$\left(\frac{dL_Y(p)}{dp}\right)_{p=q+} = \frac{u_0 + d}{\mu_Y} \neq \frac{u_0}{\mu_Y} = \left(\frac{dL_Y(p)}{dp}\right)_{p=q-}. \tag{2.1.12}$$

Consequently, $L_Y(p)$ is continuous in the point $q = F_X(a)$ but it is not differentiable and has a cusp for $p = q$.

Remark. If the transformation $u(x)$ is continuous then $d = 0$ and we obtain equality in (2.1.12) and the Lorenz curve is differentiable with the derivative $L'_Y(p) = \dfrac{y_p}{\mu_Y}$.

For progressive taxations, $u(x)$ is the post-tax income and $\dfrac{u(x)}{x}$ measures the proportion of post-tax income to the initial income and it is a monotone decreasing function satisfying the condition (i) and the Lorenz curve is increased and $F_Y(y)$ Lorenz dominates $F_X(x)$. If the taxation is a flat tax then (ii) holds and the Lorenz curve and the Gini value remain. The third case in Theorem 2.1.1 indicates that the ratio $\dfrac{u(x)}{x}$ is increasing and the Gini coefficient increases, but this case has minor practical importance. If transfer policies are studied, then the ratio $\dfrac{u(x)}{x}$ measures the relative effect of the transfer. If it decreases the relative effect of the transfer decreases with increasing income and the inequality is reduced. If $\dfrac{u(x)}{x}$ is constant, the transformation $u(x)$ is proportional to the initial income and the Lorenz curve and the Gini value remain.

2.2 Additional Properties of Lorenz Curves for Transformed Income Distributions

We follow Fellman (2012b) who considered income *X*, defined on the interval (a,b), where $0 \le a \le x \le b \le \infty$, with the distribution function $F_X(x)$, density function $f_X(x)$, mean μ_X, percentile x_p defined as $F_X(x_p) = p$ and Lorenz curve $L_X(p)$. The general formulae are

$$\mu_X = \int_a^b x\, f_X(x) dx \tag{2.2.1}$$

and

$$L_X(p) = \frac{1}{\mu_X} \int_a^{x_p} x\, f_X(x) dx, \tag{2.2.2}$$

where $a \le x_p \le b$.

We consider the transformation $Y = u(X)$, where $u(\cdot)$ is non-negative, continuous and monotone-increasing. Since the transformation can be considered as a tax ($u(x) \le x$) or a transfer policy ($u(x) \ge x$), the transformed variable is either the post-tax or the post-transfer income.

The mean and the Lorenz curve for variable *Y* are

$$\mu_Y = \int_a^b u(x) f_X(x) dx \tag{2.2.3}$$

and

$$L_X(p) = \frac{1}{\mu_X} \int_a^{x_p} u(x) f_X(x) dx, \tag{2.2.4}$$

In the following, we consider additional properties of the Lorenz curve $L_Y(p)$. If $\frac{u(x)}{x}$ is constant, then according to Theorem 1 (ii), $L_Y(p)=L_X(p)$ and the transformed Lorenz curve is identical with the initial one, a case which will be ignored.

(a) The ratio $\frac{u(x)}{x}$ is monotonically decreasing.

According to Theorem 2.1.1, $F_Y(y)$ Lorenz dominates $F_X(x)$. We introduce the values M and m such that $\lim\limits_{x\to a+}\frac{u(x)}{x}=M\le\infty$ and $\lim\limits_{x\to b-}\frac{u(x)}{x}=m\ge 0$. Consequently, $\infty\ge M\ge\frac{u(x)}{x}\ge m\ge 0$.

Let $F_X(x_p)=p$ and $F_X(x_q)=q$. Assume that $p\le q$ and that $a\le x_p\le x\le x_q\le b$ and consequently,

$$M\ge\frac{u(x_p)}{x_p}\ge\frac{u(x)}{x}\ge\frac{u(x_q)}{x_q}\ge m\,.$$

Note that points p and q are chosen arbitrarily and that the equality signs cannot be ignored because we also include the functions $\frac{u(x)}{x}$, which are not uniformly strict decreasing in the class of transformations. Hence, we have to include members for which equalities hold for almost the whole range and, in addition, sub-intervals in which strict inequalities hold can be chosen arbitrarily short and located arbitrarily within the range (a,b). If one pursues general conditions, the inequalities (2.2.8) and (2.2.9) obtained below cannot be improved. If we assume that $\frac{u(x)}{x}$ is monotonically decreasing, then $u(x)$ must be continuous, otherwise $\frac{u(x)}{x}$ should have positive jumps (Fellman, 2009).

From $\frac{u(x_p)}{x_p} \geq \frac{u(x)}{x}$ it follows that $x_p u(x) \leq x u(x_p)$. The integration over the interval $x_p \leq x \leq x_q$ yields

$$\int_{x_p}^{x_q} x_p u(x) f_X(x) dx \leq \int_{x_p}^{x_q} x u(x_p) f_X(x) dx$$

$$x_p \int_{x_p}^{x_q} u(x) f_X(x) dx \leq u(x_p) \int_{x_p}^{x_q} x f_X(x) dx$$

$$x_p \mu_Y \left(L_Y(q) - L_Y(p)\right) \leq u(x_p) \mu_X \left(L_X(q) - L_X(p)\right)$$

and

$$\left(L_Y(q) - L_Y(p)\right) \leq \frac{u(x_p)\mu_X}{x_p \mu_Y}\left(L_X(q) - L_X(p)\right). \tag{2.2.5}$$

Analogously, it follows from $\frac{u(x)}{x} \geq \frac{u(x_q)}{x_q}$ that $x_q u(x) \geq x u(x_q)$, and we obtain

$$\left(L_Y(q) - L_Y(p)\right) \geq \frac{u(x_q)\mu_X}{x_q \mu_Y}\left(L_X(q) - L_X(p)\right). \tag{2.2.6}$$

Consequently,

$$\frac{u(x_p)\mu_X}{\mu_Y x_p}\left(L_X(q) - L_X(p)\right) \geq \left(L_Y(q) - L_Y(p)\right) \geq \frac{u(x_q)\mu_X}{\mu_Y x_q}\left(L_X(q) - L_X(p)\right). \tag{2.2.7}$$

When $p \to 0$ in (2.2.7), then $L_Y(p) \to 0$, $L_X(p) \to 0$, $\frac{u(x_p)}{x_p} \to M$ and one obtains

$$\frac{M\mu_X}{\mu_Y} L_X(q) \geq L_Y(q) \geq \frac{u(x_q)\mu_X}{\mu_Y x_q} L_X(q). \tag{2.2.8}$$

The lower bound gives an evaluation of how much the Lorenz curve has increased. The upper bound is of minor interest and is commented on later.

When $q \to 1$ in (2.2.7), then $L_Y(q) \to 1$, $L_X(q) \to 1$, $\frac{u(x_q)}{x_q} \to m$ and one obtains

$$1 - \frac{m\mu_X}{\mu_Y}\left(1 - L_X(p)\right) \geq L_Y(p) \geq 1 - \frac{u(x_p)\mu_X}{\mu_Y x_p}\left(1 - L_X(p)\right).$$

In order to compare these inequalities with the inequalities in (2.2.8), we change the argument from *p* to *q*, and the inequalities are

$$\begin{aligned} 1 - \frac{m\mu_X}{\mu_Y}\left(1 - L_X(q)\right) \geq L_Y(q) \geq \\ 1 - \frac{u(x_q)\mu_X}{\mu_Y x_q}\left(1 - L_X(q)\right). \end{aligned} \tag{2.2.9}$$

The lower bound gives an evaluation of how much the Lorenz curve has increased. The upper bound is of minor interest and is discussed later.

Inequality (2.2.8) is applicable to small values and inequality (2.2.9) to large values of *q*. For small values of *q*, we consider the difference

$$D_1(q) = L_Y(q) - \frac{u(x_q)\mu_X}{\mu_Y x_q} L_X(q) \tag{2.2.10}$$

and for large q we consider the difference

$$D_2(q) = L_Y(q) - 1 + \frac{u(x_q)\mu_X}{\mu_Y x_q}\left(1 - L_X(q)\right). \qquad (2.2.11)$$

In general, $\frac{dL_Y(q)}{dq} = \frac{y_q}{\mu_Y} = \frac{u(x_q)}{\mu_Y}$ and $\frac{dL_X(q)}{dq} = \frac{x_q}{\mu_X}$.

The ratio $\frac{u(x)}{x}$ is decreasing and consequently

$$\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) = \frac{d}{dq}\left(\frac{y_q}{x_q}\right) = \frac{d}{dx_q}\left(\frac{y_q}{x_q}\right)\frac{d}{dq}\left(x_q\right) \le 0 .$$

Now we differentiate $D_1(q)$ and obtain

$$\begin{aligned}\frac{d\left(D_1(q)\right)}{dq} &= \frac{u(x_q)}{\mu_Y} - \frac{u(x_q)}{\mu_Y}\frac{\mu_X}{x_q}\frac{x_q}{\mu_X} - L_X(q)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) \\ &= -L_X(q)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) \ge 0\end{aligned}.$$

Consequently $D_1(q)$ is increasing from zero at $q = 0$ to a maximum $D_1(q_0)$ for $q = q_0$ (say).

Now we differentiate $D_2(q)$ and obtain

$$\begin{aligned}\frac{d\left(D_2(q)\right)}{dq} &= \frac{u(x_q)}{\mu_Y} - \frac{u(x_q)}{\mu_Y} + \left(1 - L_X(q)\right)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) \\ &= \left(1 - L_X(q)\right)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) \le 0 .\end{aligned}$$

Consequently $D_2(q)$ is decreasing from $D_2(q_0)$ to zero when $q \to 1$. The point q_0, at which the shift from (2.2.10) to (2.2.11) is performed, is chosen so that $D_1(q_0) = D_2(q_0)$.

Now,

$$L_Y(q_0) - 1 + \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}}\left(1 - L_X(q_0)\right) = L_Y(q_0) - \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}} L_X(q_0);$$

that is,

$$1 - \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}} = 0 \text{ and } \frac{u(x_{q_0})}{x_{q_0}} = \frac{\mu_Y}{\mu_X}.$$

Consequently,

$$D_1(q_0) = D_2(q_0) = L_Y(q_0) - L_X(q_0)$$

Since the ratio $\frac{u(x)\mu_X}{x\mu_Y}$ is decreasing, the difference $\frac{u(x_{q_0})}{x_{q_0}} - \frac{\mu_Y}{\mu_X} = 0$ shifts its sign from plus to minus at point q_0. Hemming and Keen (1983) gave the condition for Lorenz dominance that $\frac{u(x)}{x}$ crosses the $\frac{\mu_Y}{\mu_X}$ level once from above. Our results above have shown that the crossing point is q_0. The condition obtained can also be otherwise explained. If we write it as $\frac{u(x_{q_0})}{\mu_Y} = \frac{x_{q_0}}{\mu_X}$, we obtain the formula $\left.\frac{dL_Y(q)}{dq}\right|_{q=q_0} = \left.\frac{dL_X(q)}{dq}\right|_{q=q_0}$, that is, the Lorenz curves $L_Y(q)$ and $L_X(q)$ have parallel tangents and the distance $L_Y(q_0) - L_X(q_0)$ between the Lorenz curves is maximal for $q = q_0$.

We define the difference function as

$$\widetilde{D}(q) = \begin{cases} D_1(q) \text{ for } q \le q_0 \\ D_2(q) \text{ for } q > q_0 \end{cases}, \tag{2.2.12}$$

and the lower bound of $L_Y(p)$ is

$$\widetilde{L}(q) = \begin{cases} \dfrac{u(x_q)\mu_X}{\mu_Y x_q} L_X(q) & \text{for} \quad q \le q_0 \\ 1 - \dfrac{u(x_q)\mu_X}{\mu_Y x_q}\left(1 - L_X(q)\right) & \text{for} \quad q > q_0 \end{cases}. \tag{2.2.13}$$

Figure 2.2.1 shows the Lorenz curves $L_Y(q)$, $L_X(q)$, the lower bound $\widetilde{L}(q)$ and the difference $\widetilde{D}(q)$ between $L_Y(q)$ and the lower bound $\widetilde{L}(q)$.

Remarks. The variable *Y* Lorenz dominates *X*, and the upper bounds in (2.2.8) and (2.2.9) tells us nothing about the reductions in the inequality. The upper bound contains the maximum value M and one has to take it for granted that it is also inaccurate when M is finite. In addition, there may be situations in which $M = \infty$. The minimum value *m* can be zero, and in this case the upper bound is one and the obvious inequality $L_Y(p) \le 1$ is obtained.

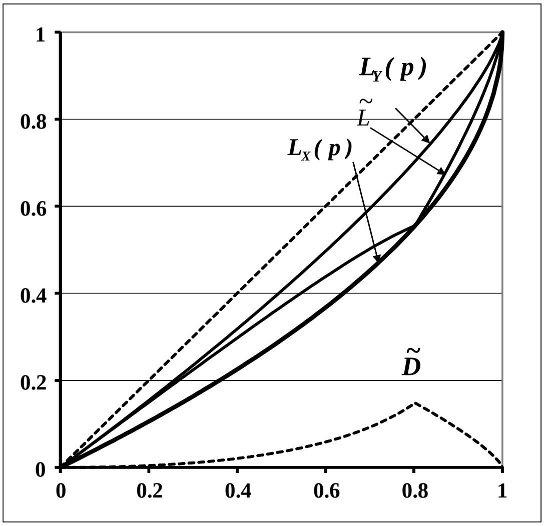

Figure 2.2.1 *(Fellman (2012b) A sketch of the Lorenz curves* $L_Y(q)$, $L_X(q)$, *the lower bound* $\tilde{L}(q)$, *and the difference* $\tilde{D}(q)$ *between* $L_Y(q)$ *and the lower bound* $\tilde{L}(q)$ *when the transformed variable Lorenz dominates the initial one.*

(b) The ratio $\frac{u(x)}{x}$ *is monotonically increasing.*

The analysis of this case follows similar traces to the earlier study and the results are analogous to our earlier results, but in this case $u(x)$ may be discontinuous. Only the inequality signs have changed their directions. We introduce the values M $(\leq \infty)$ and m (≥ 0) such that

$$\lim_{x\to a+} \frac{u(x)}{x} = m \quad \text{and} \quad \lim_{x\to b-} \frac{u(x)}{x} = M$$

and consequently $0 \leq m \leq \frac{u(x)}{x} \leq M \leq \infty$. Note, that in this case the points p and q are also chosen arbitrarily and that the equality signs cannot be ignored because we also include functions $\frac{u(x)}{x}$ which are not uniformly strictly increasing in the class of transformations. Hence, we have to include members for which equalities hold for almost the whole range and, in addition, the subintervals where strict inequalities hold can be arbitrarily short and can be located arbitrarily within the range. If one pursues general conditions, the inequalities (2.2.17) and (2.2.18) obtained below cannot be improved.

If $u(x)$ is discontinuous, the discontinuities can only be a countable number of finite positive jumps. Under such circumstances $u(x)$ is still integrable.

We use the same notations as above and assume that $F_X(x_p) = p$, $F_X(x_q) = q$, that $p \leq q$ and consequently that $x_p \leq x \leq x_q$. Now, $\frac{u(x_p)}{x_p} \leq \frac{u(x)}{x} \leq \frac{u(x_q)}{x_q}$. Consider $x_p u(x) \geq x u(x_p)$. The integration over the interval $x_p \leq x \leq x_q$ yields

$$\int_{x_p}^{x_q} x_p u(x) f_X(x) dx \geq \int_{x_p}^{x_q} x u(x_p) f_X(x) dx$$

$$x_p \int_{x_p}^{x_q} u(x) f_X(x) dx \geq u(x_p) \int_{x_p}^{x_q} x f_X(x) dx$$

$$x_p \mu_Y \left(L_Y(q) - L_Y(p)\right) \geq u(x_p) \mu_X \left(L_X(q) - L_X(p)\right)$$

and

$$\left(L_Y(q) - L_Y(p)\right) \geq \frac{u(x_p)\mu_X}{x_p \mu_Y} \left(L_X(q) - L_X(p)\right). \tag{2.2.14}$$

Analogously, if we consider $x_q u(x) \leq x u(x_q)$ we obtain

$$x_q\mu_Y\left(L_Y(q)-L_Y(p)\right)\le u(x_q)\mu_X\left(L_X(q)-L_X(p)\right)$$

and

$$\left(L_Y(q)-L_Y(p)\right)\le \frac{u(x_q)\mu_X}{x_q\mu_Y}\left(L_X(q)-L_X(p)\right). \tag{2.2.15}$$

Hence,

$$\frac{u(x_p)\mu_X}{\mu_Y x_p}\left(L_X(q)-L_X(p)\right)\le\left(L_Y(q)-L_Y(p)\right)\le$$

$$\frac{u(x_q)\mu_X}{\mu_Y x_q}\left(L_X(q)-L_X(p)\right). \tag{2.2.16}$$

When $p\to 0$ in (2.2.16), then $L_Y(p)\to 0$, $L_X(p)\to 0$, $\frac{u(x_p)}{x_p}\to m$ and one obtains

$$\frac{m\mu_X}{\mu_Y}L_X(q)\le L_Y(q)\le\frac{u(x_q)\mu_X}{\mu_Y x_q}L_X(q). \tag{2.2.17}$$

Now, the initial variable *X* Lorenz dominates the transformed *Y* and the upper bound is the interesting case.

When $q\to 1$ in (2.2.16), then $L_Y(1)\to 1$, $L_X(q)\to 1$, $\frac{u(x_q)}{x_q}\to M$ one obtains

$$1-\frac{u(x_p)\mu_X}{\mu_Y x_p}\left(1-L_X(p)\right)\ge L_Y(p)\ge 1-\frac{M\mu_X}{\mu_Y}\left(1-L_X(p)\right)$$

After a shift from *p* to *q*, we obtain

$$1-\frac{u(x_q)\mu_X}{\mu_Y x_q}\left(1-L_X(q)\right)\geq L_Y(q)\geq 1-\frac{M\mu_X}{\mu_Y}\left(1-L_X(q)\right). \qquad (2.2.18)$$

Now the upper bound is of interest. Formula (2.2.17) is applicable for small values and formula (2.2.16) for large values of q. In the following, we consider the difference between the upper bound in (2.2.17) and the Lorenz curve $L_Y(q)$, that is, for small values of q, we obtain

$$D_1(q)=\frac{u(x_q)\mu_X}{\mu_Y x_q}L_X(q)-L_Y(q). \qquad (2.2.19)$$

For large values of q, we consider the difference between the lower bound in (2.2.18) and the Lorenz curve $L_Y(q)$, that is, for small values of q, we obtain

$$D_2(q)=1-\frac{u(x_q)\mu_X}{\mu_Y x_q}\left(1-L_X(q)\right)-L_Y(q). \qquad (2.2.20)$$

In general, $\frac{dL_Y(q)}{dq}=\frac{y_q}{\mu_Y}$ and $\frac{dL_X(q)}{dq}=\frac{x_q}{\mu_X}$.

The ratio $\frac{u(x)}{x}$ is increasing and consequently,

$$\frac{d}{dq}\left(\frac{y_q}{x_q}\right)=\frac{d}{dx_q}\left(\frac{y_q}{x_q}\right)\frac{d}{dq}\left(x_q\right)\geq 0.$$

Now we differentiate $D_1(q)$ and note that $\frac{u(x_q)}{x_q}$ is increasing and obtain

$$\frac{d\left(D_1(q)\right)}{dq}=\frac{u(x_q)}{\mu_Y}\frac{\mu_X}{x_q}\frac{x_q}{\mu_X}+L_X(q)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right)-\frac{u(x_q)}{\mu_Y}=L_X(q)\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right)\geq 0.$$

Consequently $D_1(q)$ is increasing from zero to a maximum for q_0.

Now we differentiate $D_2(q)$ and obtain

$$\frac{d(D_2(q))}{dq} = +\frac{u(x_q)}{\mu_Y} - (1 - L_X(q))\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) - \frac{u(x_q)}{\mu_Y}$$
$$= -(1 - L_X(q))\frac{\mu_X}{\mu_Y}\frac{d}{dq}\left(\frac{u(x_q)}{x_q}\right) \le 0 .$$

Consequently $D_2(q)$ is decreasing from a maximum to zero. The point denoted q_0, at which the shift from $D_1(q)$ to $D_2(q)$ is performed, satisfies $D_1(q) = D_2(q)$.

Now, $1 - \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}}(1 - L_X(q_0)) - L_Y(q_0) = \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}} L_X(q_0) - L_Y(q_0)$, that is,

$$1 - \frac{u(x_{q_0})\mu_X}{\mu_Y x_{q_0}} = 0 \text{, and } \frac{u(x_{q_0})}{x_{q_0}} = \frac{\mu_Y}{\mu_X} .$$

This condition is identical with the condition in which $\frac{u(x)}{x}$ is decreasing.

Again, the condition $1 - \frac{u(x_p)\mu_X}{\mu_Y x_p} = 0$ can be written $\frac{u(x_{q_0})}{\mu_Y} = \frac{x_{q_0}}{\mu_X}$ and we obtain the formula $\frac{dL_Y(q)}{dq}\Big|_{q=q_0} = \frac{dL_X(q)}{dq}\Big|_{q=q_0}$, that is, the Lorenz curves $L_Y(q)$ and $L_X(q)$ have parallel tangents and the distance between the Lorenz curves is maximal.

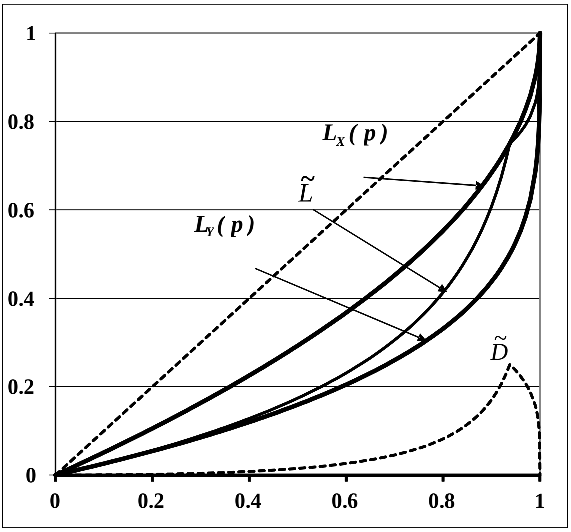

Figure 2.2.2 *(Fellman 2012b) A sketch of the Lorenz curves $L_Y(q)$, $L_X(q)$, the upper bound $\tilde{L}(q)$, and the difference $\tilde{D}(q)$ between the upper bound $\tilde{L}(q)$ and $L_Y(q)$ when the transformed variable is Lorenz dominated by the initial one.*

We define the difference function as

$$\tilde{D}(q) = \begin{cases} D_1(q) \text{ for } q \le q_0 \\ D_2(q) \text{ for } q > q_0 \end{cases}, \tag{2.2.21}$$

and the upper bound of $L_Y(q)$ is

$$\tilde{L}(q) = \begin{cases} \dfrac{u(x_q)\mu_X}{\mu_Y x_q} L_X(q) & \text{for} \quad q \le q_0 \\ 1 - \dfrac{u(x_q)\mu_X}{\mu_Y x_q}\left(1 - L_X(q)\right) & \text{for} \quad q > q_0 \end{cases}. \tag{2.2.22}$$

In Figure 2.2.2, we sketch the Lorenz curves $L_Y(q)$, $L_X(q)$, the upper bound $\tilde{L}(q)$ and the difference $\tilde{D}(q)$ between the upper bound $\tilde{L}(q)$ and $L_Y(q)$.

Now the lower bounds are of minor interest because the initial variable X Lorenz dominates Y. Note that $m = 0$ is possible in some situations and the lower bound in (2.2.17) can be zero. Note that M can be great and even $M = \infty$ is possible in some situations and the lower bound in (2.2.18) can be even negative.

Example 2.2.1 The Pareto distribution. Consider income X with the Pareto distribution $F_X(x) = 1 - x^{-\alpha}$ and $f_X(x) = \alpha x^{-\alpha-1}$, where $\alpha > 1$ and $x \geq 1$.

Now, $\mu_X = \dfrac{\alpha}{\alpha - 1}$ and the Lorenz curve $L_X(p) = 1 - (1-p)^{\frac{\alpha-1}{\alpha}}$.

From $F_X(x_p) = 1 - x_p^{-\alpha} = p$ we obtain $x_p = (1-p)^{-\frac{1}{\alpha}}$.

Let the transformation be $Y = u(x) = x^{\beta}$ ($0 < \beta < 1$) so that the function $\dfrac{u(x)}{x} = \dfrac{x^{\beta}}{x} = x^{\beta-1} = \dfrac{1}{x^{1-\beta}}$ is decreasing. We obtain $\mu_Y = \dfrac{\alpha}{\alpha - \beta}$, the Lorenz curve $L_Y(p) = 1 - (1-p)^{\frac{\alpha-\beta}{\alpha}}$ and

$$D_1(q) = 1 - \frac{1-\beta}{\alpha - 1}(1-q)^{\frac{\alpha-\beta}{\alpha}} - \frac{(\alpha-\beta)}{(\alpha-1)}(1-p)^{\frac{1-\beta}{\alpha}}$$

and

$$D_2(q) = \frac{(1-\beta)}{(\alpha-1)}(1-q)^{\frac{\alpha-\beta}{\alpha}}$$

$$\tilde{D}(q) = \begin{cases} D_1(q) = 1 - \dfrac{1-\beta}{\alpha - 1}(1-q)^{\frac{\alpha-\beta}{\alpha}} - \dfrac{(\alpha-\beta)}{(\alpha-1)}(1-p)^{\frac{1-\beta}{\alpha}} & \text{for} \quad q \leq q_0 \\ D_2(q) = \dfrac{(1-\beta)}{(\alpha-1)}(1-q)^{\frac{\alpha-\beta}{\alpha}} & \text{for} \quad q > q_0 \end{cases}$$

$$\tilde{L}(q) = \begin{cases} \dfrac{(\alpha-\beta)}{(\alpha-1)}\left((1-q)^{\frac{1-\beta}{\alpha}} - (1-q)^{\frac{\alpha-\beta}{\alpha}}\right) & \text{for} \quad q \le q_0 \\ 1 - \dfrac{(\alpha-\beta)}{(\alpha-1)}\left((1-q)^{\frac{\alpha-\beta}{\alpha}}\right) & \text{for} \quad q > q_0 \end{cases}$$

For $\beta < 1$, the ratio $\dfrac{u(x)}{x}$ is decreasing, this case being sketched in Figure 2.2.1, and if $\beta > 1$ the ratio $\dfrac{u(x)}{x}$ is increasing, this case being sketched in Figure 2.2.2.

2.3 Regional and Temporal Variation in the Income Inequality

We start with an example from Finland.

Example 2.3.1. Finland 1971-1990. We illustrate our methods using data from Finland from 1971 to 1990 (Fellman et al., 1996). The theoretical analyses of the Finnish data are presented more in detail in Chapter 5. The base *x* for taxes includes all taxable income. From this we subtract direct taxes *t* to get the base for all non-taxable benefits *b*. These are child allowances and housing subsidies. We have standardized the income variables to be comparable across households of different sizes using the OECD equivalence scale, which assigns the weight of 1.0, 0.7 and 0.5 equivalent adults to the first and additional adults and children, respectively. We show in Table 2.3.1 the estimated generalized Gini coefficients for different values of the parameter ν and the relevant income concepts. Under the actual column we see the inequality indices for original income, *x*, post-tax pre-transfer income $y = x - t$ and final income

$y+b=x-t+b$. Household disposable income per equivalent adult is equal to $x-t+b$.

We observe in Figure 2.3.1 that the Gini coefficients of original income for all v´s decrease monotonically over the period, 1971-1990, indicating decreasing income inequality.

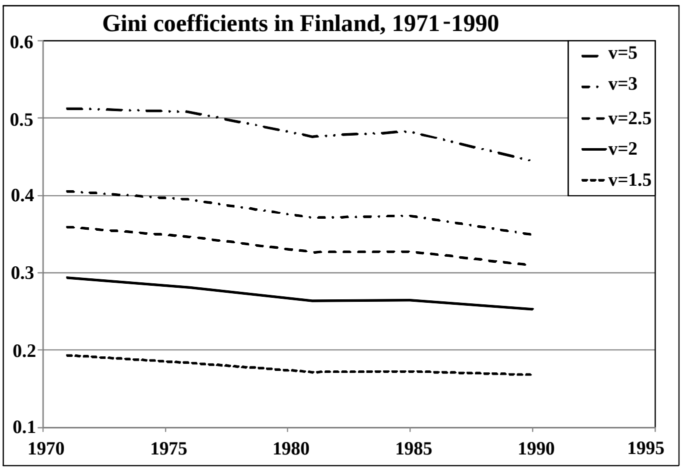

Figure 2.3.1 *Generalized Gini coefficients in Finland, 1971-1990, for different v´s.*

Eriksson and Jäntti (1997) showed that, in Finland, earnings inequality dropped dramatically between 1971 and 1975 and continued to decrease until 1985. From 1985 to 1990 there was a substantial increase in the inequality of earnings, comparable in magnitude to that found in the UK and US. Furthermore, they showed that the rise in inequality increased in Finland between 1985 and 1990 but this followed a sharp decline during the 1970s and early 1980s. The Figure 2.3.1 indicates that the conclusions given by Fellman et al. (1996) and Eriksson and Jäntti (1997) are similar for the period up to 1985 but after that they differ.

Table 2.3.1 *Inequality of income in Finland 1971-1990. Generalized Gini coefficients for pre-tax, post-tax and post-transfer income for actual incomes.*

				Actual
	Year	$\boldsymbol{x}$	$x-t$	$x-t+b$
	1971	0.193	0.176	0.173
	1976	0.183	0.168	0.165
1.5	1981	0.171	0.159	0.154
	1985	0.172	0.155	0.151
	1990	0.168	0.147	0.145
	1971	0.294	0.271	0.267
	1976	0.281	0.259	0.255
2.0	1981	0.264	0.246	0.239
	1985	0.265	0.241	0.235
	1990	0.253	0.224	0.222
	1971	0.359	0.333	0.328
	1976	0.346	0.320	0.315
2.5	1981	0.326	0.303	0.295
	1985	0.327	0.299	0.291
	1990	0.309	0.275	0.273
	1971	0.405	0.378	0.372
	1976	0.394	0.365	0.359
3.0	1981	0.371	0.346	0.335
	1985	0.373	0.342	0.332
	1990	0.349	0.313	0.311
	1971	0.512	0.483	0.476
	1976	0.507	0.472	0.465
5.0	1981	0.476	0.447	0.433
	1985	0.482	0.448	0.434
	1990	0.444	0.405	0.402

Source: Fellman et al. (1996).
Note: x is actual pre-tax income, t denotes taxes and b benefits.

Gottschalk & Smeeding (1997) compared the trends in inequality in different countries during the last decades in the 20th century. They noted marked differences. The first group consists of countries that experienced at least as large an increase in inequality as in the United States. This group includes only United

Kingdom. A second group which experienced substantial increases in inequality, but less than the United States and United Kingdom includes Canada, Australia and Israel. France, Japan, The Netherlands, Sweden and Finland form a third group with positive, but quite small changes in earnings inequality over the 1980s. Figure 2.3.1 agrees with the findings by Gottschalk and Smeeding. While even the Nordic countries experienced some increase in earnings during the 1980s, they started from very low levels, resulting from a long secular decline in inequality. Finally, Italy and Germany form a small group that experienced no measurable increase in earnings inequality during the 1980s.

Bach et al. (2009) analyzed income distributions in Germany (1932-2003) using several indicators of income inequality. They found a modest increase of the Gini coefficient, a substantial drop of median income and a remarkable growth of the income share accruing the economic elite that is the 0.001 percent of persons in the population. Their findings are supported by a relative difference between mean and median income that measures the skewness of the distribution: a rise in this measure of inequality indicates that incomes in the upper half of the distribution have increased more than the lower half.

In contrast to the findings for Finland (Fellman et al., 1996), income inequality in the United States has increased dramatically over the past 30 years. For instance, for households headed by working-age individuals, market incomes in the upper part of the distribution show an upwards trend in almost all periods since 1978, while they increased remarkably little in the middle and show large and sustained declines at the bottom during and after recessions. This is particularly true for the recent economic crisis.

Levy and Murnane (1992) presented a thorough study of the income distribution in US and discussed the variation in the inequality. For males they found that the inequality moved from stability or gradual increases in the 1970s to rapid increases in 1980s. For females they noted that annual earnings inequality moved from modest decline in the 1970s to increases in 1980s. They gave detailed interpretation of these general findings based on the variations in the composition of the labour force.

Yun (2006) studied the earnings inequality in US, 1969-1999 using different inequality measures; the ninetieth-tenth percentile log wage differential, the coefficient of variation, the Gini coefficient, the Theil index and the variance of log earnings. All measures identify an increase in the inequality. The increasing trends varied. The inequality was stable until 1980, steadily increased from 1980 to 1986, was stable again from 1987 to 1992 an increased thereafter. Autor et al. (2008) considered income inequality in US, 1963-2005. They found increasing trends and stressed that this trend was not an "episodic" one, but a continuing increase reflecting the mechanical confounding effects of changes in labour force composition. They provided an overview of the literature on U.S. wage inequality and discussed if the substantial increase since the 1980s can be considered as an episodic event or a continuous development.

Heathcote et al. (2010) conducted a systematic empirical study of cross-sectional inequality in the United States. They found a large and steady increase in wage inequality between 1967 and 2006. Taxes and transfers compress the level of income inequality, especially at the bottom of the distribution, but have little effect on the overall trend. Meyer and Sullivan (2011) found that post-tax income inequality started to increase later (in the late 1970s) than that of pre-tax income and that its increase in the 1980s occurred at a slower rate.

Analysing earlier results for US, Gottschalk and Danziger (2005) found that the development of male wage and family income inequality were largely comparable over the period 1975 to 2002. Bargain et al. (2011) noted increasing income inequality during the late 1970s and early 1980s. Furthermore, they stated that the usual approach for evaluating the role of taxation as a driver of overall inequality trends is to compare income inequality measured before and after taxes (see e.g. Gottschalk & Smeeding 1997). However, tax burdens and their impact on the income distribution are determined by both tax schedule and tax base. For instance, a given progressive income tax schedule redistributes more when the distribution of taxable incomes becomes more dispersed, and very little if everybody earns about the same (Musgrave & Thin 1948; Dardanoni & Lambert 2002). Bargain et al. (2011) concluded that main findings are as follows. The increase in post-tax income inequality was slower than that of pre-tax inequality indicating that the redistributive role of the tax system has increased over time. However, their decomposition reveals that most of this increase in redistribution was not due to the policy effect but a mechanical consequence of the rising inequality in pre-tax income.

2.4 Estimation of Gini Coefficients

Fellman (2012a) analysed the estimation of Gini coefficients using Lorenz curves. Primary income data yields the most accurate estimates of the Gini coefficient. However, the estimation must often be based on tables with grouped data or on Lorenz curves. The Lorenz curves are usually defined for five quintiles or for 10 deciles. As explained above in Section 1.1 the Gini coefficient is defined as the ratio of the area between the diagonal and the Lorenz curve and the area of the whole triangle under the diagonal. For five quintiles, the trapezium rule is the most commonly used method. However, this rule yields for every trapezium positive bias for the estimate of the area under the Lorenz curve and, consequently, the rule causes negative bias for the Gini coefficient. Simpson ́s rule is better fitted to the Lorenz curve, but demands an even number of subintervals of the same length. That is, Gini coefficients can be based on Lorenz curves given in deciles.

Various attempts have been made to produce more exact estimates. Gastwirth (1972) introduced interval estimates of the Gini coefficient in order to measure the accuracy of the estimates. Needleman ́s study (1978) starts from the trapezium estimate of the Gini coefficient G_L. He then introduces an improved upper estimate G_U. His final estimate follows the "two-thirds rule" that is $G = \frac{G_L}{3} + \frac{2G_U}{3}$. McDonald and Ransom (1981) considered the Γ density, applied Monte Carlo methods and introduced lower and upper bounds of the Gini estimates.

Golden (2008) showed how a quick approximation of the Gini coefficient can be calculated empirically, using numerical data in cumulative income quintiles. Fellman (2012a) compared different methods. He applied Simpson ́s rule and considered Lorenz curves with deciles. In addition, he used Lagrange polynomials and generalizations of Golden ́s method.

There are several different situations and, consequently, alternative analyses of Gini coefficients have to be performed. When Lorenz curves are considered, the simplest situation is that they are defined for five quintiles or for 10 deciles. In the first case, the most commonly used method is the trapezium rule. For Simpson ́s rule, the number of subintervals should be even and the intervals should have the same length. This means, for example, that Lorenz curves with 10 deciles are suitable. One has three L values for each doubled subinterval. The area under this part of the Lorenz curve is estimated so that the Lorenz curve is approximated by a parabola obtaining the same L values. Consequently, the comparison of different rules can be performed for Lorenz curves with deciles.

Following Fellman (2012a) we assume a Lorenz curve $L(p)$ with deciles. Let the observed values of the cumulative Lorenz curve be p_i and L_i for

$i = 0, 1, ..., 10$. Note that $p_i = i/10$, ($i = 0, 1, ..., 10$), that $L_0 = 0$ and that $L_{10} = 1$. According to the trapezium rule, the estimated area under the Lorenz curve is

$$\tilde{I} = ½\sum_{i=0}^{9}(L_{i+1} + L_i)(p_{i+1} - p_i) \tag{2.4.1}$$

and the estimated Gini coefficient, G_T is $1 - 2\tilde{I}$. Every trapezium yields a positive bias to the estimated area, as can be seen in Figure 2.4.1. Since the biases obtained add and no elimination of biases can be performed, the estimated Gini coefficient always has a negative bias.

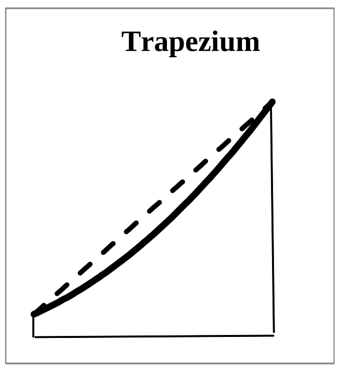

Figure 2.4.1 *A sketch showing the bias in the trapezium rule.*

Compared to the trapezium rule, Simpson´s rule gives more accurate approximations. As stressed above, Simpson´s rule demands two restrictions: the number of subintervals has to be even and the subintervals have to be of equal length. In order to obtain Simpson´s rule, the subintervals should be grouped two by two. Each doubled subinterval has three L values. The area under this part of the Lorenz curve is estimated such that a parabola obtaining the same L values approximates the Lorenz curve. Simpson´s rule obviously yields exact results for quadratic curves but, in general, this also holds for cubic curves. Assuming *2n* subintervals, the approximate area formula for a doubled interval is

$$\tilde{I}_i = \frac{1}{3n}\left(L_i + 4L_{i+1} + L_{i+2}\right),$$

the total sum is

$$\tilde{I} = \frac{1}{3n}\sum_{i=0}^{4}\left(L_{2i} + 4L_{2i+1} + L_{2i+2}\right) \tag{2.4.2}$$

and $G_S = 1 - 2\tilde{I}$.

Golden (2008) gave a detailed account of an alternative method based on Lorenz curves with quintiles. He considered p and L in percentages. The layout of the method is presented in Table 2.4.1. First he determined where the cumulative income shortfall is greatest and defined Z as the largest quintile point of the cumulative income shortfall from perfect equality divided by 100. In order to obtain the largest cumulative income shortfall he defined the transformed variable $\tilde{L}_i = L_{i-1} + 20$. This transformation, $\tilde{L}_i = L_{i-1} + 20$, indicates a search for an interval at which L_i shifts from increases faster than p_i to slower increases. For low i ´s, the transformed value $\tilde{L}_i > L_i$. Later, there is a first i value such that $\tilde{L}_i < L_i$. For this value, one finds an interval for which L is closely parallel with the diagonal, the greatest shortfall is obtained, and one defines $q = (20i - \tilde{L}_i)/100$. The estimated Gini coefficient in percentages, G_G , is $G_G = 50q(3-q)$. When this method was applied to 621 income observations, Golden (2008) noted that his approach performed better than the trapezium rule, also stressing that his method could be applied to Lorenz curves with deciles.

Fellman (2012a) generalized Golden´s method in the following way. If the Lorenz curves are given in deciles, then Golden´s transformation should be $\tilde{L}_i = L_{i-1} + 10$ and if the p_i ´s are not equidistant, then one has to define

$\tilde{L}_i = L_{i-1} + p_i - p_{i-1}$. Following Golden ́s rule, these processes have to continue until $\tilde{L}_i < L_i$. Then introduce $q = (p_i - \tilde{L}_i)/100$ and $G_G = 50q(3-q)$.

Table 2.4.1 *A layout of a Lorenz curve with deciles. Following Golden (2008), the data is given in percentages. The transformed* $\tilde{L}_{20i} = L_{20i-20} + 20$ *values appear in the text.*

i	**0**	**1**	**2**	**3**	**4**	**5**
p_i	0	20	40	60	80	100
L_i	$L_0 = 0$	$L\ L_{20}$	L_{40}	L_{60}	L_{80}	$L_{100} = 100$
$\tilde{L}_i$	$\tilde{L}_0 = 0$	$\tilde{L}_{20}$	$\tilde{L}_{40}$	$\tilde{L}_{60}$	$\tilde{L}_{80}$	$\tilde{L}_{100}$

In many empirical situations, the income distribution $F(x)$ is given in grouped tables. If the mean of or total incomes in the groups are known, the cumulative distribution can be considered as a Lorenz curve, but the subintervals are usually not of constant length. The trapezium rule holds, but it still yields a positive bias for the area and negative bias for the Gini coefficient.

An obviously better alternative is to approximate the Lorenz curve with Lagrange ́s interpolation (Berrut & Trefethen, 2004). Lagrange polynomials of the second degree can be considered as a generalisation of Simpson ́s rule and do not demand subintervals of equal length, but the number of subintervals should still be even. The polynomials obtained have to be integrated in order to yield approximate areas and Gini coefficients. If the subintervals are of the same length, the Lagrange polynomial method is identical with Simpson ́s rule.

Fellman (2012a) applied the Lagrange interpolation of second degree. However, he had to assume an even number of subintervals. Now the Lagrange polynomial is

$$L(p) = \sum_{i=0}^{n-1} \left(L_{2i} \frac{(p - p_{2i+1})(p - p_{2i+2})}{(p_{2i} - p_{2i+1})(p_{2i} - p_{2i+2})} + L_{2i+1} \frac{(p - p_{2i+2})(p - p_{2i})}{(p_{2i+1} - p_{2i+2})(p_{2i+1} - p_{2i})} + L_{2i+2} \frac{(p - p_{2i+1})(p - p_{2i})}{(p_{2i+2} - p_{2i+1})(p_{2i+2} - p_{2i})} \right) \tag{2.4.3}$$

This approximate polynomial must be integrated in order to obtain an estimate of the area under the Lorenz curve.

The comparison between different estimation methods is in general difficult to perform. These difficulties are mainly caused by the fact that the true Gini coefficient is unknown, but sometimes, where more detailed studies have already resulted in very accurate estimates, the comparisons are possible. Some authors (e.g., Gastwirth, 1972; Mehran, 1975; McDonald & Ransom, 1981; Rigo, 1985; Giorgi & Pallini, 1987) have introduced interval estimates, but these are often rather broad and it is still difficult to identify the best method. Such comparison problems are eliminated if the numerical estimations are applied to theoretical distributions.

Needleman (1978) stated that as the Lorenz curve is convex, the trapezium approximation is always greater than the actual area under the curve, so that the estimate based on this approximation is always less than the actual value of the coefficient. Furthermore, he noted that most authors using the trapezium approximation indicate that they are aware of the bias involved, but either assume the error so small as to be insignificant, or else use a large number of intervals in the belief, usually justified, that the bias will then be negligible. McDonald and Ransom (1981) introduced lower and upper bounds of the Gini estimates. In order to estimate the bounds of the Gini coefficient estimates, they considered the income to have a Γ density, that is, $g(y) = \frac{\beta^{\alpha} y^{\alpha-1} e^{-y\beta}}{\Gamma(\alpha)}$ with corresponding $G = \frac{\Gamma(\alpha + ?}{\Gamma(\alpha+1)\sqrt{\pi}}$ and $\mu = \alpha / \beta$ and applied Monte Carlo methods.

In order to perform comparisons between the estimated and theoretical Gini coefficients Fellman (2012a) analysed classes of theoretical Lorenz curves with varying Gini coefficients. He compared Gini estimates for the Pareto distributions. If one defines the Pareto distribution as $F(x) = 1 - x^{-\alpha}$, where $x \geq 1$ and $\alpha > 1$. Then the frequency function is $f(x) = \alpha x^{\alpha-1}$, the mean is $\mu = \frac{\alpha}{\alpha - 1}$, the quantiles are $x_p = \left(\frac{1}{1-p}\right)^{\frac{\alpha-1}{\alpha}}$, the Lorenz curve $L(p) = 1 - (1-p)^{\frac{\alpha-1}{\alpha}}$ and the Gini coefficient $G = \frac{1}{2\alpha - 1}$. Fellman considered $1.5 \leq \alpha \leq 5.0$, then the Gini coefficient satisfies the inequalities $0.111 \leq G \leq 0.500$. This G interval corresponds to the most common Gini coefficients. Fellman's results appear in Table 2.4.2 and Figure 2.4.2. Note that Simpson ́s and Golden ́s rules yield similar accuracy, but the trapezium rule shows the largest errors for all levels of Gini coefficients. This theoretical study indicates that Golden ́s rule is not uniformly better than the trapezium rule.

Gastwirth (1972) presents interval estimations of the Gini coefficient. The exact Gini estimate on Current Population Surveys (CPS) income data for 1968 was computed by Tepping, his result being 0.4014. Gastwirth ́s Table 2 shows Tepping ́s data grouped into a 10 subgroup Lorenz curve. He compares his Gini interval estimates with Tepping ́s finding. Gastwirth (1972) considers a minimum of restrictive conditions, obtaining the interval $0.3883 < G < 0.4083$. Mehran (1975) suggests an alternative estimation method, obtaining the interval estimate $0.3883 < G < 0.4087$. The grouping limits are not equidistant and one cannot apply Simpson´s rule. Applying the trapezium rule yields *0.3883* and the negative bias is apparent. The Lagrange rule yields *0.4033* and the modification of the Golden rule yields the rather inaccurate estimate *0.3740*.

Table 2.4.2 *(Fellman, 2012a). The estimation of the Gini coefficient applied to the Lorenz curve for the Pareto distributions. Note that the estimated Gini coefficients according to the trapezium rule are inaccurate and show negative biases. Simpson´s and Golden´s rules yield similar accuracy, but Golden is best for large Gini values.*

		Estimates			Error	
G	**Trapezium**	**Simpson**	**Golden**	**Trapezium**	**Simpson**	**Golden**
11.11	10.858	11.044	11.104	-0.253	-0.067	-0.008
12.50	12.206	12.419	12.529	-0.294	-0.081	0.029
14.29	13.935	14.185	14.370	-0.350	-0.101	0.084
16.67	16.235	16.535	16.833	-0.431	-0.132	0.166
20.00	19.442	19.816	20.291	-0.558	-0.184	0.291
25.00	24.223	24.717	25.476	-0.777	-0.283	0.476
33.33	32.102	32.820	34.026	-1.232	-0.513	0.693
50.00	47.481	48.730	50.317	-2.519	-1.270	0.317

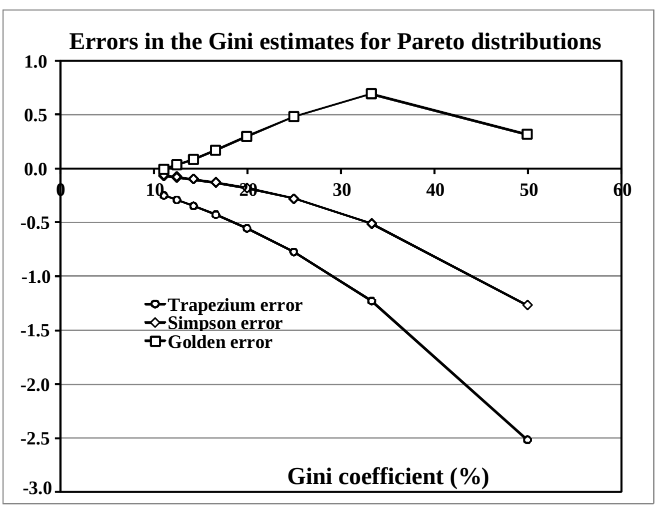

Figure 2.4.2 *Estimation errors in the Gini coefficients estimated by the trapezium, Simpson, and Golden rules. Note that Simpson´s and Golden´s rules yield similar accuracy, but the trapezium rule shows the largest errors (Fellman, 2012a).*

Lorenzen (1980) presents information about the total distribution of income for households in Germany in 1973 in his *Tabelle 2*. The Gini coefficient calculated by Lorenzen is based on data pooled in his *Tabelle 3*, which yielded *0.30*. Using Lorenzen´s *Tabelle 3*, Fellman performed a comparison of the estimates obtained based on the trapezium rule and the Lagrange rule. The available empirical data cannot yield a comparison of the accuracy of the two methods. The estimated Gini coefficient according to the trapezium rule shows negative biases compared to Lorenzen´s result, being *0.2920*. The Lagrange interpolation yields the estimate *0.3486* and the modified Golden method 0.3002.

This study indicates that the biased trapezium rule is almost always inferior and shows negative biases. No method however is uniformly optimal. Note that Simpson´s and Golden´s rules yield similar accuracy. Golden´s method is usually of medium quality, but its accuracy fluctuates.

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