4

Transferring

In Chapter one and two, we have introduced the central properties of income distributions and the methods how to analyse income distributions and redistributions. We have also given example how to estimate distributions and concentration measures in empirical data. In Chapter 3 we have presented the effect of taxation on the income distribution and inequality. In this chapter we apply the theory in order to analyse the effects of transfer policies.

4.1 The Class of Transfer Policies

In this section we present the results of a study of a class of transfer (benefit) policies. Below we compare some results concerning transfer policies with our earlier results concerning tax policies. Consider an initial income distribution, defined in Chapter 3, with the distribution function $F_X(x)$, density function $f_X(x)$, mean μ_X, Lorenz curve $L_X(p)$, the Gini coefficient G_X, generalized Gini coefficient $G_X(\nu)$ (Yitzhaki, 1983) and welfare index W_X (Sen, 1973). Following Fellman (1995, 2001) and Fellman et al. (1996, 1999), we consider the class of transfer policies characterized by the transformation $Y = h(X)$, where $h(\cdot)$ is non-negative, monotone-increasing and continuous with the properties

$$\text{H: } \begin{cases} h(x) \geq x \\ E(Y) = \mu_X + \rho \end{cases}. \tag{4.1.1}$$

The function $h(x)$ is income including government cash transfers associated with the original income x and ρ is the mean transfer. The scenario pursued here can apply as well to an income policy; in that case $h(x)$ is income after an increase according to the policy. The monotonity of $h(x)$ indicates that the initial income order remains fixed. The first formula in (4.1.1) is obvious and the second indicates that the class ***H*** of transfer policies is constrained to distribute a given amount of benefit (ρ).

The class ***H*** contains both progressive and non-progressive policies and is therefore an adaptive tool for inequality and welfare studies. It is necessary already at this stage to point out that the transformed incomes corresponding to the policies in ***H*** do not have a Lorenz ordering.

We present some general results analogous to the results holding for the class of tax policies in Chapter 3. For details, see Fellman (1995 and 2001). Consider a set of arbitrary policies $h_i(x), (i=1,\dots,k)$, belonging to ***H***. Following the analyses in Section 3.1 we obtain that the transformation

$$h_\theta(x) = \sum_{i=1}^{k} \theta_i h_i(x) \quad \theta_i \geq 0 \quad (i=1,\dots,k) \quad \sum_{i=1}^{k} \theta_i = 1, \tag{4.1.2}$$

also belongs to ***H*** because

$$h_\theta(x) = \sum_{i=1}^{k} \theta_i h_i(x) \geq \sum_{i=1}^{k} \theta_i x = x \sum_{i=1}^{k} \theta_i = x, \tag{4.1.3}$$

and

$$E\left(\sum_{i=1}^{k} \theta_i h_i(x)\right) = \sum_{i=1}^{k} \theta_i E\left(h_i(X)\right) = \sum_{i=1}^{k} \theta_i \left(\mu_X + \rho\right) = \mu_X + \rho . \tag{4.1.4}$$

Denote the corresponding Lorenz curves by $L_i(p), (i=1,\dots,k)$ and $L_\theta(p)$ and the corresponding Gini coefficients by $G_i, (i=1,\dots,k)$ and G_θ, then $h_\theta(x)$ has the Lorenz curve

$$L_\theta(p) = \sum_{i=1}^{k} \theta_i L_i(p) \tag{4.1.5}$$

and the Gini coefficient

$$G_\theta = \sum_{i=1}^{k} \theta_i G_i \ . \qquad (4.1.6)$$

Consequently, we obtain a theorem which is analogous to Theorem 3.1.1.

Theorem 4.1.1. (Fellman, 1995 and 2001) The class **H** and the classes of Lorenz curves and Gini coefficients corresponding to the policies in **H** are convex.

In order to obtain the range of the policies, we consider the member

$$h_0(x) = \begin{cases} b_0 & x \le b_0 \\ x & x > b_0 \end{cases}, \qquad (4.1.7)$$

i.e. all incomes below the level b_0 are raised up to b_0 and all incomes above this level remain as they were. The policy (4.1.7) is an example of the minimum salary policy.

For an arbitrary value of b,

$$\begin{aligned} E(h_0(x)) &= \int_0^b b f_X(x)dx + \int_b^\infty x f_X(x)dx \\ &= bF_X(b) + \mu_X - \mu_X L_X(F_X(b))e(b) \end{aligned} .$$

Now, $e(0) = \mu_X$, and we obtain

$$\lim_{b\to\infty}(e(b)) = \lim_{b\to\infty} bF_X(b) + \mu_X - \mu_X \lim_{b\to\infty} L_X(F_X(b)) = \infty .$$

The derivative $e'(b) = F_X(b) > 0$ and $e(b)$ is monotone increasing. Consequently, there is a unique b_0 such that $E(h_0(X)) = \mu_X + \rho$. The policy (4.1.7) yields an income distribution that Lorenz dominates all the income distributions for the class ***H*** and has the Lorenz curve (Fellman, 1995, 2001; Fellman et al., 1996, 1999)

$$L_0(p) = \begin{cases} \dfrac{b_0}{\mu_X + \rho} p & p \le q_0 \\ \dfrac{b_0}{\mu_X + \rho} q_0 + \dfrac{\mu_X}{\mu_X + \rho}(L_X(p) - L_X(q_0)) & p > q_0 \end{cases} \tag{4.1.8}$$

where $F_X(b_0) = q_0$.

The Lorenz curve $L_0(p)$ has continuous derivative because for $p \le q_0$, $L_0'(p) = \dfrac{b_0}{\mu_X + \rho}$ and for $p > q_0$, $L_0'(p) = \dfrac{\mu_X}{\mu_X + \rho} L'_X(p) = \dfrac{\mu_X}{\mu_X + \rho} \dfrac{x_p}{\mu_X} = \dfrac{x_p}{\mu_X + \rho}$ which converges towards $\dfrac{b_0}{\mu_X + \rho}$ when $p \to q_0$.

The function $L_C(p) = \dfrac{b_0}{\mu_X + \rho} q_0 + \dfrac{\mu_X}{\mu_X + \rho}(L_X(p) - L_X(q_0))$ is not a Lorenz curve, but it is convex and has $L_T(p) = \dfrac{b_0}{\mu_X + \rho} p$ as tangent in the point q_0. Hence, $L_C(p) \ge L_0(p)$ for all $p \in [0,1]$. From $L_0(1) = 1$, we obtain $b_0 q_0 - \mu_X L_X(q_0) = \rho$ and the corresponding Gini coefficient is

$$G_0 \ge 1 - 2\int_0^1 L_C(p)dp = 1 - 2\frac{\mu_X}{\mu_X + \rho}\int_0^1 L_X(p)dp - 2\frac{b_0 q_0}{\mu_X + \rho} + 2\frac{\mu_X}{\mu_X + \rho} L_X(q_0)$$

$$G_X + \frac{\rho}{\mu_X + \rho}(1 - G_X) - 2\frac{\rho}{\mu_X + \rho} =$$

$$G_X - \frac{\rho}{\mu_X + \rho}(1 + G_X).$$

Consequently, G_0 satisfies the inequality

$$G_0 \geq G_X - \frac{\rho}{\mu_X + \rho}(1 + G_X). \quad (4.1.9)$$

This bound is obviously a lower bound of the Gini coefficient of all policies in $\boldsymbol{H}$.

For the transfer policies, a lowest Lorenz curve cannot be found, but we can attain arbitrarily closely an inferior Lorenz curve (Fellman, 2001). Consider the sequence of transfer policies

$$\boldsymbol{H}_S: \; h_i(x) = \begin{cases} x & x < x_i \\ x + k_i(x - x_i) & x \geq x_i \end{cases} \quad i = 1, 2, \ldots \quad (4.1.10)$$

These policies give no benefits to the poorest part of the population ($x < x_i$), but positive benefits to the richest part ($x \geq x_i$). We construct the sequence so that $\boldsymbol{H}_S \subseteq \boldsymbol{H}$ and that their Lorenz curves converge towards an inferior Lorenz curve. If we define $k_i\,(>0)$ so that $\int_{x_i}^{\infty} k_i(x - x_i) f_X(x)dx = \rho$, then every $h_i(x)$ is continuous and monotone increasing, $h_i(x) \geq x$ and $E(h_i(X)) = \mu_X + \rho$. Hence, $\boldsymbol{H}_S \subseteq \boldsymbol{H}$ and the corresponding Lorenz curve is (Fellman, 1995 and 2001):

$$L_i(p) = \begin{cases} \dfrac{\mu_X}{\mu_X + \rho} L_X(p) & p < q_i \\ \dfrac{\mu_X}{\mu_X + \rho} L_X(q_i) + \dfrac{\rho}{\mu_X + \rho} \dfrac{\mu_X\left(L_X(p) - L_X(q_i)\right) - x_i(p - q_i)}{\mu_X\left(1 - L_X(q_i)\right) - x_i(1 - q_i)} & p \geq q_i \end{cases}, \quad (4.1.11)$$

where $F_X(x_i) = q_i$.

If we choose the sequence $i = 1,2,...$ so that $x_i \to \infty, q_i \to 1$ and hence, $k_i \to \infty$ in (4.1.10) we obtain a limit Lorenz curve

$$L_\infty(p) = \begin{cases} \dfrac{\mu_X}{\mu_X + \rho} L_X(p) & p < 1 \\ 1 & p = 1 \end{cases} . \qquad (4.1.12)$$

This Lorenz curve has no well-defined income distribution.[4] It does not correspond to a member of the class ***H*** but we can come arbitrarily close to it by choosing q_i in (4.1.11) arbitrarily close to *1*, that is, the proportion of benefit-receivers tends towards zero. We can prove:

Theorem 4.1.2. (Fellman, 1995). The Lorenz curve $L_\infty(p)$ is inferior to the Lorenz curves for the class **H**.

Proof. Choose an arbitrary policy $h(x)$ in ***H***. We can evaluate its Lorenz curve in the following way:

$$L_h(p) = \frac{1}{\mu_X + \rho} \int_0^{x_p} h(x) f_X(x)dx \geq \frac{1}{\mu_X + \rho} \int_0^{x_p} x f_X(x)dx = \frac{\mu_X}{\mu_X + \rho} L_X(p) = L_\infty(p)$$

This inequality holds for all $p \leq 1$.

Figure 4.1.1 gives examples of the Lorenz curves $L_X(p)$, $L_0(p)$ and $L_\infty(p)$. The Lorenz curves (4.1.8) and (4.1.12) define the semi-closed region of attainable Lorenz curves (Figure 4.1.1).

[4]Vaguely speaking the limit inferior Lorenz curve in (4.1.12) corresponds to a policy, which gives all benefits to the richest income-receiver.

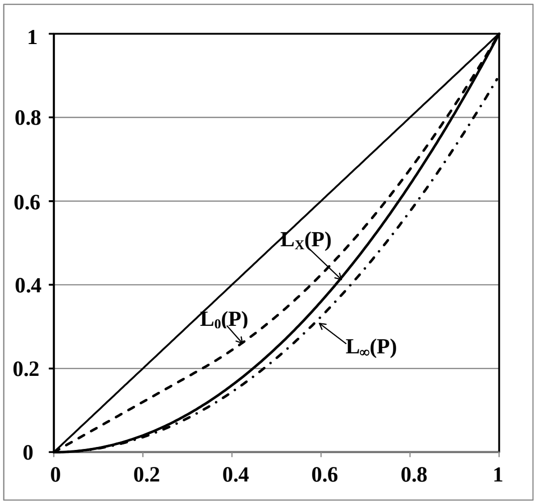

Figure 4.1.1 *The Lorenz curves $L_X(p)$, $L_0(p)$ and $L_\infty(p)$. The region between the extreme Lorenz curves is the region for attainable Lorenz curves (Fellman, 2001).*

The Gini coefficient G_∞ corresponding to the Lorenz curve (4.1.12) is

$$G_\infty = 1 - 2\int_0^1 L_\infty(p)dp = 1 - 2\frac{\mu_X}{\mu_X + \rho}\int_0^1 L_X(p)dp =$$

$$1 - \frac{\mu_X}{\mu_X + \rho}(1 - G_X) = G_X + \frac{\rho}{\mu_X + \rho}(1 - G_X). \qquad (4.1.13)$$

The final formula (4.1.13) is an exact equality and is an upper bound for all Gini coefficients for the policies in ***H***. However, it does not correspond to any member of the class ***H***. From Theorem 4.1.2 and the convergence of the policies in $\boldsymbol{H}_S$ it follows that (4.1.13) is a supremum of the Gini coefficients belonging to ***H***. For the Gini coefficient, the generalized Gini coefficient and the welfare index we obtain the bounds (Fellman, 1995):

$$G_X - \frac{\rho}{\mu_X + \rho}(1 + G_X) \le G_h \le G_X + \frac{\rho}{\mu_X + \rho}(1 - G_X), \qquad (4.1.14)$$

$$G_X(\nu)-\frac{\rho}{\mu_X+\rho}(\nu-1+G_X(\nu))\le G_h(\nu)\le G_X(\nu)+\frac{\rho}{\mu_X+\rho}(1-G_X(\nu)) \quad (4.1.15)$$

and

$$W_X \le W_h \le W_X + 2\rho . \quad (4.1.16)$$

From the deduction of the bounds in (4.1.14), (4.1.15) and (4.1.16) it follows that the formulae hold for arbitrary income distributions. For a specific income distribution the lower bounds in (4.1.14) and (4.1.15) and the upper bound in (4.1.16) can be sharpened and the accuracy of the bounds was discussed in detail in Fellman (1995, 2001). The strength of the bounds is that they are independent of the distribution $F_X(x)$ and depend only on the basic quantities G_X, μ_X and ρ. In addition, the formulae obtained, are simple functions of these quantities. If $\rho \to 0$ then both the upper and lower bounds in (4.1.14), (4.1.15) and (4.1.16) converge towards G_X , $G_X(\nu)$ and W_X , respectively, indicating that the approximations performed do not cause any "bias".

The bounds presented cannot be used as approximate formulae for a specific policy in ***H***. The central role of these intervals is that they define limits for the attainable index values and hence give indications of the redistributive effect of the class of transfer policies. C.f. with the analysis of the formulae (3.1.20), (3.1.21) and (3.1.22) for the tax policies in Section 3.1.

Every Gini coefficient belonging to the semi-closed interval $\tilde{G} \in [G_0, G_\infty)$ and every point within the semi-closed region limited by the Lorenz curves $L_\infty(p)$ (excluded) and $L_0(p)$ (included) is attainable by a member of the class ***H***. These results can be given in following theorems:

Theorem 4.1.3. (Fellman, 1995)**.** There is a member of the class **H** with a prescribed Gini coefficient $\tilde{G} \in [G_0, G_\infty)$.

Proof. Choose q_i in (4.1.11) so that the corresponding member of the sequence (4.1.10) has a Gini coefficient G_i which exceeds $L_h(p)$. Construct a member of the class H as a linear combination of (4.1.7) and this member of (4.1.11). We get $\tilde{G} = \theta G_0 + (1-\theta) G_i$ and the prescribed value of the Gini coefficient is obtained for

$$\tilde{\theta} = \frac{\tilde{G} - G_0}{G_i - G_0}. \tag{4.1.17}$$

This means that there exists a policy that gives a post-transfer income distribution with the Gini coefficient $\tilde{G}$.

Remark. Theorem 4.1.3 says that there exists at least one member of the class ***H*** that results in a post-transfer income distribution with a prescribed Gini coefficient within the closed interval $[G_0, G_\infty)$. In general, this policy is not unique, but the extreme coefficient G_0 is attainable only by the extreme policy.

We can also prove:

Theorem 4.1.4. (Fellman, 1995). There is a member of the class **H** that satisfies the condition $L_h(\tilde{p}) = \tilde{l}$, where $\tilde{l} \in (L_\infty(\tilde{p}), L_0(\tilde{p})]$.

Proof. Choose a member from the set (4.1.10) such that its corresponding Lorenz curve $L_i(p)$ satisfies the inequality $L_i(\tilde{p}) < \tilde{l}$. The solution to Theorem 4.1.4 can be constructed by a linear combination of the policy (4.1.7) and the chosen member of (4.1.10). The prescribed condition is obtained for $\tilde{l} = \theta L_i(\tilde{p}) + (1-\theta) L_0(\tilde{p})$ and

$$\tilde{\theta} = \frac{\tilde{l} - L_0(\tilde{p})}{L_1(\tilde{p}) - L_0(\tilde{p})}. \qquad (4.1.18)$$

Every point within the closed region, limited by the Lorenz curves $L_0(p)$ and $L_\infty(p)$, is attainable by a Lorenz curve corresponding to a member of the class ***H***. This means that there exists a policy that gives a post-transfer income distribution such that the lowest proportion $\tilde{p}$ of income receivers receives exactly the proportion $\tilde{l}$ of the total amount of post-transfer income. Within the class ***H***, the solution discussed in Theorem 4.1.4 is in general not unique.

Remark. In fact, Theorem 4.1.4 can be generalised to the whole region $\tilde{l} \in [L_\infty(\tilde{p}), L_0(\tilde{p})]$. If the given point is located on the lower border, that is $L_h(\tilde{p}) = L_\infty(\tilde{p})$, then the solution is a member of the sub-set $\boldsymbol{H}_S$ under the restriction $q_i \geq \tilde{p}$.

Consider the Lorenz curve $L_X(p)$ and a Lorenz curve $L_h(p)$ for an arbitrary member $Y = h(X)$. According to the general theory, we have

$$L'_X(p) = \frac{x_p}{\mu_X}$$

and

$$L'_h(p) = \frac{y_p}{\mu_X + \rho}.$$

Now, $y_p = h(x_p)$ and $y_p \geq x_p$, indicating that *Y* stochastically dominates *X*. Furthermore, we obtain

$$\frac{L'_h(p)}{L'_X(p)} \geq \frac{\mu_X}{\mu_X + \rho}. \qquad (4.1.19)$$

This is a necessary restriction on feasible Lorenz curves for members of the class ***H***. In general, there may be Lorenz curves between the extreme ones that do not correspond to policies in the class ***H***. The inequality (4.1.19) indicates also that the Lorenz curve for the transformed variables cannot differ markedly from the Lorenz curve of *X*. This is especially notable for small values of ρ (ρ / μ_X). For the extreme policy (4.1.7) and for the sequence of policies in (4.1.10), equality in the formula (4.1.19) is obtained for whole subintervals; $q_0 \leq p \leq 1$ and $0 \leq p \leq q_i$, respectively. For the inferior Lorenz curve (4.1.12), equality holds within the semi-closed interval $0 \leq p < 1$. These properties stress the optimality of the extreme policies. In the next section we give necessary and sufficient conditions that a given Lorenz curve corresponds to a transfer policy belonging to the class *H*.

4.2 Attainable Lorenz Curves

Above we noted that among post-transfer income distributions there exist distributions with given coefficients and distributions whose Lorenz curves have given, prescribed co-ordinates (p, l). However, we also stressed that every Lorenz curve within the admissible region is not necessarily attainable. Now, we derive necessary and sufficient conditions that a given Lorenz curve corresponds to a transfer policy belonging to the given class ***H***. A similar study has been performed for tax policies in Chapter 3 and in Fellman (2001, 2002).

In general, let U and V be non-negative stochastic variables having the distributions $F_U(u)$ and $F_V(v)$, the means μ_U and μ_V and the Lorenz curves $L_U(p)$ and $L_V(p)$, respectively. Stochastic dominance of first, second and third order can be defined by alternative equivalent–conditions. Some of these are given by Maasoumi and Heshmati (2000). (cf. also Davidson and Duclos 2000, Klonner, 2000 and Zheng, 2000). Using our notations, the Maasoumi-Heshmati (2000) definition of stochastic dominance of first order is (c.f. Definition 3.2.1).

Definition 4.2.1. *The variable U First Order Stochastic Dominates V if and only if any one of the following equivalent conditions holds:*

i. $E[g(U)] \geq E[g(V)]$ *for all increasing functions g.*

ii. $F_U(u) \leq F_V(u)$ *for all u.*

iii. $u_p \geq v_p$ *for all p* $(0 < p < 1)$.

We can prove the following theorem (c.f. Lemma 3.2.1).

Theorem 4.2.1. Let U and V be non-negative stochastic variables having the distributions $F_U(u)$ and $F_V(v)$, the means μ_U and μ_V and the Lorenz curves $L_U(p)$ and $L_V(p)$, respectively, then the conditions:

(i) U stochastically dominates V.

(ii) $F_U(v) \leq F_V(v)$ for all v.

(iii) $u_p \geq v_p$ for all p $(0 < p < 1)$.

(iv) $\dfrac{L'_V(p)}{L'_U(p)} \leq \dfrac{\mu_U}{\mu_V}$ for all p $(0 < p < 1)$.

are equivalent.

Proof. The equivalence between (i), (ii) and (iii) is given in Definition 4.2.1. Now, we only have to prove the equivalence between (iii) and (iv) (say). The connection between (iii) and (iv) are the general formulae

$$L'_U(p) = \frac{u_p}{\mu_U} \text{ and } L'_V(p) = \frac{v_p}{\mu_V}. \quad (4.2.1)$$

a) Assume that (iii) holds. Now, using (4.2.1) we obtain

$$1 \geq \frac{v_p}{u_p} = \frac{\mu_V L'_V(p)}{\mu_U L'_U(p)}$$

and the condition (iv) is obtained.

b) Assume that (iv) holds. Now

$$\frac{\mu_U}{\mu_V} \geq \frac{L'_V(p)}{L'_U(p)} = \left(\frac{v_p}{\mu_V}\right)\left(\frac{u_p}{\mu_U}\right)^{-1}$$

$$1 \geq \frac{v_p}{u_p},$$

$$u_p \geq v_p$$

and the proof is completed.

Remark. The condition (iv) in Theorem 4.2.1, being equivalent with (i), (ii) and (iii), indicates that (iv) is a criterion for stochastic dominance of first order between two positive stochastic variables. This criterion was also presented in Fellman (2003) and in Chapter 3.

In Section 4.1 we have noted that stochastic dominance of first order is a necessary condition that a transformed distribution is a post-transfer income distribution corresponding to a policy of the class ***H***. In the following we obtain sufficient conditions. Our results can be given in the following theorem, which is analogous to Theorem 3.2.1.

Theorem 4.2.2. Consider a Lorenz curve $\overline{L}_Y(p)$ and a corresponding stochastic variable Y with the distribution $\overline{F}_Y(y)$ and the mean $(\mu_X + \rho)$. Then the necessary and sufficient condition that the Lorenz curve $\overline{L}_Y(p)$ is an attainable Lorenz curve of a member of **H**, $\overline{h}(x) = (\mu_X + \rho)\overline{L}'(F_X(x))$ being the member and $\overline{F}_Y(y)$ being the corresponding post-transfer distribution, is that one of the following equivalent conditions hold:

i. Y stochastically dominates X.

ii. $F_X(x) \geq \overline{F}_Y(x)$ for all x.

iii. $y_p \geq x_p$ for all p ($0 < p < 1$) or.

iv. $\dfrac{\overline{L}'_Y(p)}{L'_X(p)} \geq \dfrac{\mu_X}{\mu_X + \rho}$.

Proof. Assume that the presumptive post-transfer income distribution is $\overline{F}_Y(y)$ with the mean $(\mu_X + \rho)$. We introduce the quantiles x_p and y_p, where $F_X(x_p) = p$ and $\overline{F}_Y(y_p) = p$. These quantiles can also be defined as $F_X^{-1}(p) = x_p$ and $\overline{F}_Y^{-1}(p) = y_p$. In Section 4.1 we noted that

$$y_p \geq x_p \text{ for all p } (0 < p < 1) \tag{4.2.2}$$

is a necessary condition for $\overline{F}_Y(y)$ to be an attainable post-transfer income distribution. From (4.2.2) it follows that

$$\overline{F}_Y(x_p) \le \overline{F}_Y(y_p) = p = F_X(x_p) \text{ for all p } (0 < p < 1).$$

The condition

$$F_X(x) \ge \overline{F}_Y(x) \text{ for all x} \quad (4.2.3)$$

being equivalent with (4.2.2), is also a necessary condition that the post-transfer income distribution corresponds to a transfer policy belonging to ***H***. From formula (4.2.3) we obtain

$$\overline{F}_Y^{-1}(F_X(x)) \ge \overline{F}_Y^{-1}(\overline{F}_Y(x)) = x \text{ for all x.} \quad (4.2.4)$$

In the following we prove that the condition that the distribution satisfies (4.2.3) is sufficient, that is, the distribution is a post-transfer income distribution for a member of the class ***H***. Consequently, the equivalent conditions (i) and (iii) are also sufficient.

Consider a distribution with mean $(\mu_X + \rho)$ satisfying (4.2.3). According to the definition of a distribution function we have

$$P(Y \le y) = \overline{F}_Y(y).$$

The cumulative distribution function is monotone increasing and

$$P(\overline{F}_Y(Y) \le \overline{F}_Y(y)) = \overline{F}_Y(y). \quad (4.2.5)$$

If $Z = \overline{F}_Y(Y)$ and $z = \overline{F}_Y(y)$, then $Y = \overline{F}_Y^{-1}(Z)$, $y = \overline{F}_Y^{-1}(z)$ and

$$P(Z \le z) = z. \quad (4.2.6)$$

The transformed variable $Z = \overline{F}_Y(Y)$ is uniformly distributed over the interval (*0*, *1*) and (4.2.6) is an old well-known result. Consider the initial distribution $F_X(x)$. Then

$$z = P(Z \leq z) = P(F_X^{-1}(Z) \leq F_X^{-1}(z)). \quad (4.2.7)$$

Let $X = F_X^{-1}(Z)$ and $x = F_X^{-1}(z)$ then $Z = F_X(X)$ and $z = F_X(x)$.

Now,

$$y = \overline{F}_Y^{-1}(z) = \overline{F}_Y^{-1}(F_X(x)) = \overline{h}(x) \text{ (say).} \quad (4.2.8)$$

Hence $\overline{h}(x)$ is continuous and monotone increasing. In addition, from (4.2.4) follows that $\overline{h}(x)$ satisfies the condition

$$\overline{h}(x) \geq x \quad (4.2.9)$$

and $\overline{h}(x)$ belongs to ***H*** and the distribution $\overline{F}_Y(y)$, having the mean $(\mu_X + \rho)$, corresponds to a policy belonging to the class ***H*** and the sufficiency is obtained.

Let us now introduce Lorenz curves and obtain the conditions that a specific Lorenz curve (and the corresponding distribution $\overline{F}_Y(y)$) can be attained by a member of the class ***H***. Let us consider an arbitrary Lorenz curve $\overline{L}(p)$ with the conditions:

i. $\overline{L}(p)$ has a continuous derivative of the first order ($\overline{L}'(p)$).

ii. $\lim_{p \to 1}(1 - p)\overline{L}'(p) = 0$.

These conditions imply that the corresponding distribution $\overline{F}_Y(y) = M(\frac{y}{\mu})$, where $M(\cdot)$ is the inverse function to $\overline{L}'(p)$ is continuous and has a finite mean (Fellman, 1976, 1980). In those papers, it was assumed that the second derivative exists but this condition is not necessary in this context. In general, when the Lorenz curve $\overline{L}(p)$ and the mean are given, the corresponding income distribution is unique.

Consider the distribution $\overline{F}_Y(y)$ with the mean $\mu_X + \rho$. We have $\overline{L}'(p) = \frac{y_p}{\mu_X + \rho}$ and $y_p = \overline{F}_Y^{-1}(p) = (\mu_X + \rho)\overline{L}'(p)$. From these formulae it follows that $\overline{h}(x) = (\mu_X + \rho)\overline{L}'(F_X(x))$. Hence, the condition

$$\overline{h}(x) = (\mu_X + \rho)\overline{L}'(F_X(x)) \geq x \tag{4.2.10}$$

is a necessary condition for attainability. On the other hand let us assume that (4.2.10) holds. Let $\overline{F}_Y(y)$ be the distribution, which corresponds to $\overline{L}(p)$ and has the mean $\mu_X + \rho$. Then

$$x_p \leq (\mu_X + \rho)\overline{L}'(F_X(x_p)) = (\mu_X + \rho)\overline{L}'(p) = (\mu_X + \rho)\overline{L}'(\overline{F}_Y(y_p)) = y_p,$$

and $y_p \geq x_p$ for all p ($0 < p < 1$). Consequently, (4.2.10) is also sufficient and the proof is completed.

The content in Theorem 4.2.2 indicates that every stochastic variable *Y*, which stochastically dominates *X* and whose mean is $\mu_X + \rho$ and whose Lorenz curve belongs to the semi-closed Lorenz region, is attainable by a policy belonging to ***H***. The condition (iv) implies after integrations that

$$(\mu_X + \rho)\bar{L}(p) \geq \mu_X L_X(p)\,. \tag{4.2.11}$$

indicating *Generalized Lorenz Dominance* (GLD). The integration step from $\bar{L}'(p) \geq \dfrac{\mu_X}{(\mu_X + \rho)} L'_X(p)$ given in (iv) in Theorem 4.2.2 to the condition (4.2.11) is not reversible. Consequently, GLD is only a necessary condition, or otherwise expressed, stochastic dominance implies GLD (cf. Lambert 2001 p. 49).

4.3 Discontinuous Transfer Policies with a Given Lorenz Curve

In earlier papers we have studied classes ***H*** of continuous transfer policies, defined in (4.1.1) and below in (4.3.1). In this section we consider an expanded class ***H****, containing discontinuous policies, defined below in (4.3.2)**,** and generalize the results holding for class ***H*** to class ***H****. A realistic transformation describing a general transfer policy must be continuous, but we also are prepared to consider situations where discontinuous policies are plausible.

For class ***H*** we have obtained supreme and inferior Lorenz curves $L = L_0(p)$ in (4.1.8) and $L = L_\infty(p)$ in (4.1.12). In addition, we have proved that there are policies belonging to ***H*** with given Gini coefficients or Lorenz curves passing through given points in the (p, L) plane. The necessary and sufficient conditions under which a given Lorenz curve $\bar{L}(p)$ corresponds to a member of class ***H*** of transfer policies are equivalent to the condition that the transformed variable $\bar{Y} = h(X)$ stochastically dominates the initial variable *X*.

The notations in this section will be similar to those in our earlier sections. Let the income be X with the distribution function $F_X(x)$, density function $f_X(x)$, mean μ_X and Lorenz curve $L_X(p)$. The basic formulae are

$$\mu_X = \int_0^{\infty} x\, f_X(x)dx$$

and

$$L_X(p) = \frac{1}{\mu_X}\int_0^{x_p} x\, f_X(x)dx,$$

where $F_X(x_p) = p$.

We introduce the transformation $Y = h(X)$, where $h(\cdot)$ is non-negative and monotone increasing. Since the transformation can be considered as a tax ($h(x) \le x$) or a transfer ($h(x) \ge x$) policy, the transformed variable Y is either the post-tax or post-transfer income. The mean and the Lorenz curve for variable Y are

$$\mu_Y = \int_0^{\infty} h(x) f_X(x)dx$$

and

$$L_Y(p) = \frac{1}{\mu_Y}\int_0^{x_p} h(x) f_X(x)dx.$$

A general theorem concerning Lorenz dominance is (Fellman, 1976; Jakobsson, 1976; Kakwani, 1977 and also given in Theorem 1.4.1):

Theorem 4.3.1. Let X be a non-negative, random variable with distribution $F_X(x)$, mean μ_X and Lorenz curve $L_X(p)$. Let $h(x)$ be a non-negative, monotone-increasing function; let $Y = h(X)$ and let $E(Y) = \mu_Y$ exist. The Lorenz curve $L_Y(p)$ of Y exists, and the following results hold:

i. $L_Y(p) \geq L_X(p)$ if $\dfrac{h(x)}{x}$ is monotone decreasing.

ii. $L_Y(p) = L_X(p)$ if $\dfrac{h(x)}{x}$ is constant.

iii. $L_Y(p) \leq L_X(p)$ if $\dfrac{h(x)}{x}$ is monotone increasing.

Recently, Egghe (2009) returned to Theorem 4.3.1 and gave a new proof. In addition, he showed that the theorem is not true for the dual transformation.

Fellman (1980, 2003) introduced the class of transfer policies

$$\text{H: } \begin{cases} h(x) \geq x \\ E(h(X)) = \mu_X + \rho \end{cases}, \tag{4.3.1}$$

where $h(x)$ is non-negative, monotone increasing and continuous.

This class was considered in Section 4.1. Now we modify this class allowing $h(x)$ to be discontinuous and define

$$\boldsymbol{H^*}\text{: } \begin{cases} h(x) \geq x \\ E(h(X)) = \mu_X + \rho \end{cases}, \tag{4.3.2}$$

where $h(x)$ is non-negative and monotone increasing.

If $h(x)$ is discontinuous and satisfies Theorem 4.1.1 and the transformation should result in an increasing transformed variable with finite mean then the discontinuities can only consist of finite positive jumps and the number of jumps can be assumed to be finite or countable. (Fellman, 2009).

Consider an optimal policy which Lorenz dominates all policies in ***H****. According to Theorem 4.1.1, $\frac{h(x)}{x}$ must be monotonically decreasing. Consequently, it must be continuous because if $h(x)$ has a discontinuity point, then the ratio $\frac{h(x)}{x}$ has a positive jump and cannot be monotonously decreasing. Consequently, although class ***H****, in comparison with the initial class ***H***, also contains discontinuous policies, the policy

$$h_0(x) = \begin{cases} b_0 & x \le b_0 \\ x & x > b_0 \end{cases}, \tag{4.3.3}$$

being optimal among all continuous policies, is still optimal for the class ***H****. It has the Lorenz curve

$$L_0(p) = \begin{cases} \frac{b_0}{\mu_Y + \rho} p & p \le q_0 \\ \frac{b_0}{\mu_Y + \rho} q_0 + \frac{\mu_X}{\mu_Y + \rho}\left(L_X(p) - L_X(q_0)\right) & p > q_0 \end{cases}, \tag{4.3.4}$$

where $q_0 = F_X(b_0)$. The inferior Lorenz curve presented in Section 4.1 can be obtained from the sequence (Fellman, 2001)

$$\boldsymbol{H_S}\text{:}\ h_i(x) = \begin{cases} x & x < x_i \\ x + k_i(x - x_i) & x \ge x_i \end{cases} \quad (i = 1, 2, \ldots). \tag{4.3.5}$$

Define k_i (>0) so that

$$\int_{x_i}^{\infty} k_i(x - x_i) f_X(x) dx = \rho,$$

then $h_i(x)$ is continuous and monotone increasing, $h_i(x) \geq x$ and $E(h_i(X)) = \mu_X + \rho$. Hence, $\boldsymbol{H_s} \subseteq \boldsymbol{H} \subseteq \boldsymbol{H^*}$, and the corresponding Lorenz curve is (Fellman, 2001)

$$L_{hi}(p) = \begin{cases} \dfrac{\mu_X}{\mu_X + \rho} L_X(p) & p < q_i \\ \dfrac{\mu_X}{\mu_X + \rho} L_X(q_i) \\ + \dfrac{\rho}{\mu_X + \rho} \dfrac{\mu_X(L_X(p) - L_X(q_i)) - x_i(p - q_i)}{\mu_X(1 - L_X(q_i)) - x_i(1 - q_i)} & p \geq q_i \end{cases}. \tag{4.3.6}$$

If we choose the sequence $i = 1, 2, \ldots$ so that $x_i \to \infty$, $q_i \to 1$, and hence, $k_i \to \infty$ in (6), we obtain a limit Lorenz curve

$$L_\infty(p) = \begin{cases} \dfrac{\mu_X}{\mu_X + \rho} L_X(p) & p < 1 \\ 1 & p = 1 \end{cases}. \tag{4.3.7}$$

Independently of the existence of discontinuity points we can still prove (Fellman, 2009).

Theorem 4.3.2. The Lorenz curve $L_\infty(p)$ is inferior to the Lorenz curves for class **H***.

Proof. Consider an arbitrary, continuous or discontinuous policy $h(x)$ in $\boldsymbol{H^*}$ with Lorenz curve $L_h(p)$. Using the condition $h(x) \geq x$, we can evaluate $L_\infty(p)$ in the following way:

$$L_h(p) = \frac{1}{\mu_X + \rho} \int_0^{x_p} h(x) f_X(x) dx \geq \frac{1}{\mu_X + \rho} \int_0^{x_p} x f_X(x) dx = \frac{\mu_X}{\mu_X + \rho} L_X(p) = L_\infty(p) . \quad (4.3.8)$$

This inequality holds for all $0 \leq p \leq 1$.

In addition, $\boldsymbol{H} \subseteq \boldsymbol{H^*}$, and consequently, there exist policies belonging to ***H**** with given Gini coefficients or Lorenz curves passing through given points in the (p, L) plane. Hence, class ***H**** of transfer policies containing discontinuous policies satisfies the same properties as the initial class ***H*** discussed in Fellman (1980) and Fellman (2001). Figure 4.3.1 includes the Lorenz curves $L_X(p)$, $L_0(p)$ and $L_\infty(p)$. The figure also shows a Lorenz curve, $L_Y(p)$ with a cusp corresponding to a discontinuous transformation.

In Fellman (2003) and in Section 4.2 we obtained necessary and sufficient conditions under which a given differentiable Lorenz curve $\overline{L}(p)$ corresponds to a member of a given class of transfer policies. These conditions are equivalent to the condition that the transformed variable $Y = h(X)$ stochastically dominates the initial variable *X*.

Now we generalize the results, including the classes with discontinuous transformations. A discontinuous transformation $h(x)$ can only have a countable number of positive finite steps, and every jump in the transformation $h(x)$ results in a cusp in the Lorenz curve.

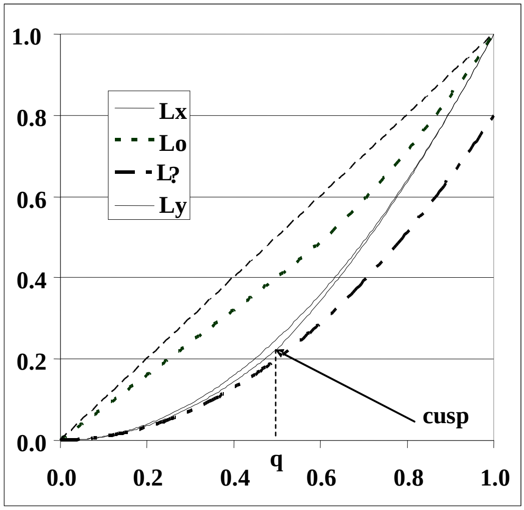

Figure 4.3.1 *A sketch of the Lorenz curves* $L_X(p)$ *and* $L_Y(p)$, *when* $h(x)$ *is discontinuous for* $x = a$ *and* $q = F_X(a)$. *Note the cusp of* $L_Y(p)$ *at the point* $p = q$. *The figure also includes the maximum and minimum Lorenz curves* $L_0(p)$ *and* $L_\infty(p)$ *for the transfer policies in* ***H**** *(Fellman, 2003).*

Consider a Lorenz curve $\overline{L}(p)$ which is convex and differentiable everywhere with the exception of a countable number of cusps. More general Lorenz curves cannot be considered. The corresponding distribution is $\overline{F}_Y(y) = M(\frac{y}{\mu})$, in which $M(\cdot)$ is the inverse function to $\overline{L}'(p)$ and Y is assumed to have the mean $\mu = \mu_X + \rho$ (Fellman, 1980). If $\overline{L}(p)$ has a cusp, then the derivative $\overline{L}'(p)$ and the function $M(\cdot)$ have positive jumps.

In general, when Lorenz curve $\overline{L}(p)$ and the mean of the corresponding distribution are given, the income distribution is unique (see Chapter 1). Now, we prove that the conditions obtained earlier still hold for class ***H****, that is, we will characterize attainable Lorenz curves even though they are not universally differentiable.

Fellman (2003) has noted that stochastic dominance of first order is a necessary condition for a transformed distribution to be a post-transfer income distribution corresponding to a policy of class ***H*** in (4.3.1). The result is given in:

Theorem 4.3.3. Consider Lorenz curve $\overline{L}_Y(p)$ and a corresponding stochastic variable Y with the distribution $\overline{F}_Y(y)$ and the mean $(\mu_X+\rho)$. Then the necessary and sufficient condition that the Lorenz curve $\overline{L}_Y(p)$ is an attainable Lorenz curve of a member of **H***,

$$\overline{h}(x)=(\mu_X+\rho)\overline{L}'(F_X(x)),$$

being the member and $\overline{F}_Y(y)$ being the corresponding post-transfer distribution, is that one of the following equivalent conditions hold:

i. Y stochastically dominates X.

ii. $F_X(x)\ge\overline{F}_Y(x)$ for all x.

iii. $y_p\ge x_p$ for all p ($0<p<1$) or.

iv. $\dfrac{\overline{L}'(p)}{L_X'(p)}\ge\dfrac{\mu_X}{\mu_X+\rho}$.

When we prove this theorem for class ***H****, we have to show that $y_p\ge x_p$ holds for distribution $\overline{F}_Y(y)$. The proof of Theorem 4.2.2 given earlier in Fellman (2003) can be applied as such for $\overline{h}(x)$ wherever it is continuous, but the discontinuity points need special attention. Class ***H**** of transfer policies containing discontinuous policies satisfies the same properties as the initial class discussed in Fellman (1980) and Fellman (2003), and we obtain the transformation

$$y = \bar{h}(x) = (\mu_X + \rho)\bar{L}'(F_X(x)).$$

If $\bar{L}(p)$ has a cusp for $p = q$, then $\bar{h}(x)$ has a jump for x_q. Consider a neighbourhood $x_q - h < x < x_q + h$, where x_q is the only discontinuity point of $\bar{h}(x)$, and choose a $\delta > 0$ so small that $x_q - h < x_{q-\delta} < x_{q+\delta} < x_q + h$.

Let $\lim_{\delta \to 0+} h(x_{q-\delta}) = y_{q-}$ and $\lim_{\delta \to 0+} h(x_{q+\delta}) = y_{q+} > y_{q-}$.

Now, the transformation $\bar{h}(x)$ is continuous for all $\delta > 0$, and

$$y_{q-\delta} = (\mu_X + \rho)\bar{L}'(F_X(x_{q-\delta})) \geq x_{q-\delta}.$$

When $\delta \to 0_+$, the inequality holds for the limits, and we obtain $y_{q-} \geq x_q$. Similarly, we obtain

$$y_{q+\delta} = (\mu_X + \rho)\bar{L}'(F_X(x_{q+\delta})) \geq x_{q+\delta},$$

and when $\delta \to 0_+$, the inequality holds for the limits, and we obtain $y_{q+} \geq x_q$. Hence, $y_p \geq x_p$ for all *p*, and $Y = \bar{h}(X)$ stochastically dominates the initial variable *X*.

4.4 Discussion

In this chapter we have studied the effects of transfer policies. A realistic transformation describing a general transfer policy must be continuous. The generalized class ***H**** of transfer policies containing discontinuous policies satisfies the same properties as the initial class discussed in Fellman (1980) and Fellman (2003). The theory presented here is obviously applicable in connection with other income redistributive studies such that the discontinuity can be assumed to be realistic. If the problem is reductions in taxation, then the tax reduction for a taxpayer can be considered as a new benefit (Fellman, 2001). Consequently, the class of transfer policies ***H**** can be used for comparisons between different tax-reducing policies. If transfers are increased, the effect of increases on a receiver can also be studied through transfer policies ***H****. In general, such changes may be mixtures of several different components and discontinuity cannot be excluded, and the continuity assumption can be dropped. One general result is still that continuity is a necessary condition if one expects that income inequality should remain or be reduced. Analogously, tax increases and transfer reductions can be considered as new tax policies (Fellman, 2001).

Empirical applications of the optimal policies among a class of tax policies and the class of transfer policies have been discussed in Fellman et al. (1999), where we developed "optimal yardsticks" to gauge the effectiveness of given real tax and transfer policies in reducing inequality. We return to these problems in the next chapter.

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