# 3

# Taxation

In Chapter one and Chapter two, we have introduced the central properties of income distributions and the methods how to analyse income distributions and redistributions. We have also given example how to estimate distributions and concentration measures in empirical data. In this chapter we apply the methods on the effects of taxation policies.

## 3.1 A Class of Tax Policies

Following Fellman (2001) we consider a pre-tax income *X*, assumed given, with the distribution function $F_X(x)$, density function $f_X(x)$, mean $\mu_X$, Lorenz curve $L_X(p)$ and the Gini coefficient $G_X$. Now, we consider a class of tax policies characterized by the transformation $Y = u(X)$ where $u(\cdot)$ is non-negative, monotone increasing and continuous with the properties

$$\mathbf{U}: \begin{cases} u(x) \le x \\ u'(x) \le 1 \\ E(u(X)) = \mu_X - \tau \end{cases} . \qquad (3.1.1)$$

The function $u(x)$ is the post-tax income associated with the pre-tax income $x$ and $\tau$ is the mean tax. The monotonicity of $u(x)$ indicates that the internal order of the incomes remains the same after taxation. The taxation reduces the income and consequently, the first condition in (3.1.1) is obvious. The second guarantees that also the taxes increase monotonically with increasing initial income *x*, and the third indicates that the different tax policies yield the same total amount of taxes when applied to the given pre-tax incomes. In order to give a more realistic definition of the class of tax policies, Fellman (2001) introduced the restriction $u'(x) \le 1$. Earlier in Fellman (1995) and in Fellman et al. (1996, 1999), this restriction was not assumed. Therefore, some of the results in those studies differ slightly from the results in later papers and in this study.

The class ***U*** of tax policies contains both progressive and non-progressive policies and is therefore an adaptive tool for inequality and welfare studies. The policies in ***U*** do not have a Lorenz ordering. Accordingly, the Lorenz curves corresponding to post-tax distributions generated by members of ***U*** may intersect.

Assume set of arbitrary policies $u_i(x)$, ( $i=1,\dots,k$ ), belonging to ***U***. Consider their linear combination

$$u_\theta(x)=\sum_{i=1}^{k}\theta_i u_i(x) \quad \theta_i \geq 0\,(\,i=1,\dots,k\,,\,)\ \sum_{i=1}^{K}\theta_i=1. \tag{3.1.2}$$

We obtain

$$u_\theta(x)=\sum_{i=1}^{k}\theta_i u_i(x)\leq\sum_{i=1}^{k}\theta_i x=x\sum_{i=1}^{k}\theta_i=x\,, \tag{3.1.3}$$

$$u'_\theta(x)=\sum_{i=1}^{k}\theta_i u'_i(x)\leq\sum_{i=1}^{k}\theta_i=1 \tag{3.1.4}$$

and

$$E\left(\sum_{i=1}^{k}\theta_i u_i(X)\right)=\sum_{i=1}^{k}\theta_i E\big(u_i(X)\big)=\sum_{i=1}^{k}\theta_i\big(\mu_X-\tau\big)=\mu_X-\tau\,. \tag{3.1.5}$$

Hence, $u_\theta(x)$ belongs to ***U*** and ***U*** is a convex class of policies.

Denote by $L_i(p)$ the Lorenz curves, $G_i$ the Gini coefficients corresponding to the policies $u_i(x)$ ( $i=1,\dots,k$ ). From the fact that integration is a linear operator we obtain the Lorenz curve $L_\theta(p)$ and the Gini coefficient $G_\theta$

$$L_\theta(p)=\frac{1}{\mu-\tau}\int_0^{x_p}\left(\sum_{i=1}^{k}\theta_i u_i(x)\right)f_X(x)dx=$$

$$\sum_{i=1}^{k}\theta_i \frac{1}{\mu-\tau}\int_0^{x_p} u_i(x) f_X(x)dx = \sum_{i=1}^{k}\theta_i L_i(p) \tag{3.1.6}$$

and

$$G_\theta = 1 - 2\int_0^1 L_\theta(p)dp = \sum_{i=1}^{k}\theta_i \int_0^1 L_i(p)dp = \sum_{i=1}^{k}\theta_i G_i \,. \tag{3.1.7}$$

Conversely, if we consider a Lorenz curve satisfying (3.1.6) and (3.1.7) it corresponds to a policy of the form (3.1.2) and belongs to ***U***. Hence, the classes of Lorenz curves and of Gini coefficients are also convex and we can summarize all results in:

**Theorem 3.1.1.** The class **U** and the classes of Lorenz curves and of Gini coefficients corresponding to the policies in **U** are convex.

Now we study the class (3.1.1) of policies in more detail. First we analyse a policy which serves as a benchmark for the members of policies. Consider

$$u_0(x) = \begin{cases} x & x \le a_0 \\ a_0 & x > a_0 \end{cases}, \tag{3.1.8}$$

that means that for incomes $x \le a_0$ there is no tax and for $x > a_0$ the tax is $x - a_0$ so that the post-tax is constantly equal to $a_0$.

We prove that there is a unique value $a_0$ such that $E(u_0(X)) = \mu_X - \tau$ and consequently, the corresponding policy belongs to ***U***. For an arbitrary *a* we obtain

$$E(u_0(X)) = \int_0^a x f_X(x)dx + \int_a^\infty a f_X(x)dx =$$

$$\mu_X L_X\left(F_X(a)\right)+a\left(1-F_X(a)\right) \tag{3.1.9}$$

The function

$$e(a)=\mu_X L_X\left(F_X(a)\right)+a\left(1-F_X(a)\right) \tag{3.1.10}$$

starts from the value $e(0)=0$ and has the derivative

$$e'(a)=\mu_X \frac{a}{\mu_X} f_X(a)+1-F_X(a)-af_X(a)=1-F_X(a)\geq 0\,. \tag{3.1.11}$$

From the fact that the mean $\mu_X$ exists, it follows that

$$\lim_{a\to\infty} e(a)=\lim_{a\to\infty}\mu_X L_X\left(F_X(a)\right)+\lim_{a\to\infty} a\left(1-F_X(a)\right)=\mu_X$$

because

$$\lim_{a\to\infty}\mu_X L_X\left(F_X(a)\right)=\mu_X$$

and

$$0\leq\lim_{a\to\infty} a\left(1-F_X(a)\right)=\lim_{a\to\infty} a\int_a^\infty f_X(x)dx=\lim_{a\to\infty}\int_a^\infty af_X(x)dx\leq\lim_{a\to\infty}\int_a^\infty xf_X(x)dx=0\,.$$

Hence, the function $e(a)$ is continuous and monotone increasing from $e(0)=0$ to $e(\infty)=\mu_X$ and consequently, there exists a unique $a_0$ such that $E\left(u_0(X)\right)=e(a_0)=\mu_X-\tau$. This value $a_0$ satisfies the inequality $a_0\geq\mu_X-\tau$ (with equality if and only if $F_X\left(a_0\right)=0$). For this value of $a_0$ the tax policy $u_0(x)$ belongs to **U**.

Define $p_0=F_X(a_0)$. For $p\leq p_0$,

$$L_0(p) = \frac{1}{\mu_X - \tau}\int_0^{x_p} x f_X(x)dx = \frac{\mu_X}{\mu_X - \tau} L_X(p)$$

and for $p > p_0$

$$L_0(p) = \frac{\mu_X}{\mu_X - \tau} L_X(p_0) + \frac{1}{\mu_X - \tau}\int_{x_{p0}}^{x_p} a f_X(x)dx = \frac{\mu_X}{\mu_X - \tau} L_X(p_0) + \frac{a_0}{\mu_X - \tau}(p - p_0).$$

Hence, the corresponding Lorenz curve is

$$\mathrm{L}_0(p) = \begin{cases} \dfrac{\mu_X}{\mu_X - \tau} L_X(p) & \mathrm{p} \le p_0 \\ \dfrac{\mu_X}{\mu_X - \tau} L_X(p_0) + \dfrac{a_0}{\mu_X - \tau}(p - p_0) & \mathrm{p} > p_0 \end{cases}. \qquad (3.1.12)$$

By definition given above, $F_X(a_0) = p_0$ and $x_{p_0} = a_0$. In the point $p = p_0$ the derivative to the left is

$$\frac{\mu_X}{\mu_X - \tau} L'_X(p_0) = \frac{x_{p_0}}{\mu_X - \tau} = \frac{a_0}{\mu_X - \tau}$$

and to the right is

$$\frac{a_0}{\mu_X - \tau}.$$

Therefore the derivative exists also in the point $p = p_0$ and the Lorenz curve (3.1.12) has a continuous derivative within the interval (0, 1).

Consider an arbitrary transformation $u(x)$ with the properties (3.1.1). Then according to Figure 3.1.1, $u(x) \le u_0(x) = x$ for $x \le a_0$.

From the fact that the function $u(x)$ is an increasing function it follows that there exists a unique $x* \geq a_0$ such that $u(x) < a_0$ for $x < x*$ and $u(x) \geq a_0$ for $x \geq x*$. Hence,

$$u(x) < u_0(x) \text{ for } x < x* \text{ and } u(x) \geq u_0(x) \text{ for } x \geq x*.$$

The difference

$$D(p) = \frac{1}{\mu_X - \tau} \int_0^{x_p} \left(u_0(x) - u(x)\right) f_X(x) dx, \qquad (3.1.13)$$

where $F_X(x_p) = p$, increases monotonically from zero to a maximum for $p* = F_X(x*)$, where after it decreases monotonically to zero. Hence, $u_0(x)$ generates a post-tax income distribution that Lorenz dominates all tax policies of the given class ***U*** (Fellman, 1995, 2001; Fellman et al., 1996, 1999). Furthermore, it also Lorenz dominate the flat tax policy $\hat{u}(x) = \frac{\mu_X - \tau}{\mu_X} x$, whose mean is $\mu_X - \tau$ and Lorenz curve $L_X(p)$. Consequently, $L_0(p) \geq L_X(p)$ and $u_0(x)$ Lorenz dominates the initial income variable *X*.

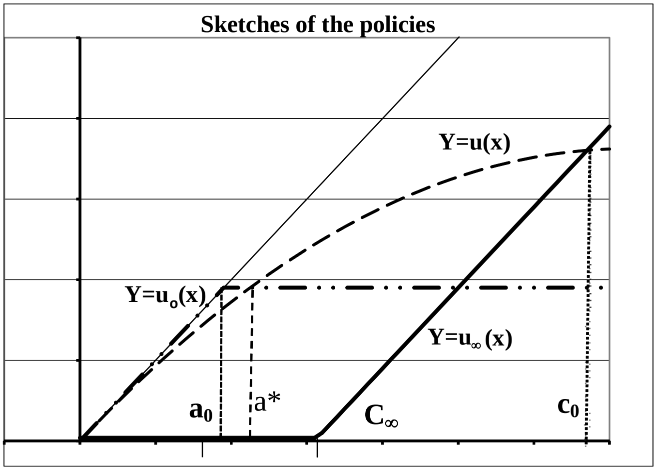


***Figure 3.1.1*** *Sketches of the two extreme tax policies:* $Y=u_o(X)$ *and* $Y=u_\infty(X)$*, and an arbitrary policy* $Y=u(x)$ *(after Fellman, 2001, 2002, 2014).*

Let $G_0$ be the Gini coefficient corresponding to $u_0(x)$. We obtain

$$\begin{aligned} G_0 &= 1-2\int_0^1 L_0(p)dp \geq 1-2\int_0^1 L_X(p)dp \\ &= 1-2\frac{\mu_X}{\mu_X-\tau}(1-G_X) = G_X - \frac{\tau}{\mu_X-\tau}(1-G_X) \end{aligned} . \qquad (3.1.14)$$

The policy (3.1.8) Lorenz dominates the class ***U*** and therefore we obtain that the lower bound $G_X - \frac{\tau}{\mu_X-\tau}(1-G_X)$ in (3.1.14) is a lower bound of the Gini coefficients of all policies in ***U***.

Consider another extreme policy

$$u_\infty(x) = \begin{cases} 0 & x < c_\infty \\ x - c_\infty & x \geq c_\infty \end{cases} . \qquad (3.1.15)$$

It takes everything from the poorest whose income is below $c_\infty$ and a constant amount $c_\infty$ from the riches whose income is greater than $c_\infty$. A sketch of $u_\infty(x)$ is also presented in Figure 3.1.1. Below we prove that there exists a value $c_\infty$ such that $u_\infty(x)$ satisfies the condition (3.1.1) and belongs to ***U***. For an arbitrary $c$ we obtain

$$\begin{aligned}E(u_\infty(X)) &= \int_0^\infty u_\infty(x) f_X(x)dx = \int_c^\infty (x-c) f_X(x)dx \\ &= \int_c^\infty xf_X(x)dx - \int_c^\infty cf_X(x)dx \\ &= \int_0^c xf_X(x)dx + \int_c^\infty xf_X(x)dx - \int_0^c xf_X(x)dx - \int_c^\infty cf_X(x)dx \\ &= \mu_X(1-L_X(F_X(c)) - c(1-F_X(c))\end{aligned} \quad . (3.1.16)$$

Consider the function $e(c) = \mu_X\left(1-L_X(F_X(c)\right) - c\left(1-F_X(c)\right)$. From the fact that $\mu_X$ exists then $e(0) = \mu_X$ and

$$\lim_{c\to\infty}\left(e(c)\right) = \lim_{c\to\infty}\left(\mu_X\left(1-L_X(F_X(c))\right) - \lim_{c\to\infty}\left(c\left(1-F_X(c)\right)\right) = \right.$$

$$-\lim_{c\to\infty} c\left(1-F_X(c)\right) = \lim_{c\to\infty} c\int_c^\infty f_X(x)dx \le \lim_{c\to\infty}\int_c^\infty xf_X(x)dx = 0 .$$

Consider the derivative $e'(c)$. Now

$$e'(c) = -\mu_X \frac{c}{\mu_X} f_X(c) - \left(1-F_X(c)\right) + cf_X(c) = -\left(1-F_X(c)\right) < 0$$

and $e(c)$ is monotone decreasing from $\mu_X$ to zero. Hence, there exists a unique value $c_\infty$ such that

$$e(c_\infty) = \mu_X\left(1-L_X(F_X(c_\infty)\right) - c_\infty\left(1-F_X(c_\infty)\right) = \mu_X - \tau$$

and the policy (3.1.15) belongs to U.

Let $F_X(c_\infty) = r_\infty$ and we get the condition

$$e(c_\infty) = \mu_X\left(1 - L_X(r_\infty)\right) - c_\infty\left(1 - r_\infty\right) = \mu_X - \tau$$

or equivalently

$$\mu_X L_X(r_\infty) + c_\infty\left(1 - r_\infty\right) = \tau\,. \tag{3.1.17}$$

The corresponding Lorenz curve is

$$L_\infty(p) = \begin{cases} 0 & p < r_\infty \\ \dfrac{\mu_X}{\mu_X - \tau}\left(L_X(p) - L_X(r_\infty)\right) - \dfrac{c_\infty(p - r_\infty)}{\mu_X - \tau} & p \geq r_\infty \end{cases}. \qquad 3.1.18)$$

This Lorenz curve is continuous and has a derivative in the whole interval $(0,1)$ because in the point $p = r_\infty$ the derivative to the left is zero and to the right is $\dfrac{\mu_X}{\mu_X - \tau}\dfrac{c_\infty}{\mu_X} - \dfrac{c_\infty}{\mu_X - \tau} = 0$. Figure 3.1.1 gives examples of the extreme policies $Y = u_0(X)$ and $Y = u_\infty(X)$, and an arbitrary policy $Y = u(x)$.

A sketch of the Lorenz curves $L_X(p)$, $L_0(p)$, and $L_\infty(p)$ is given in Figure 3.1.2.

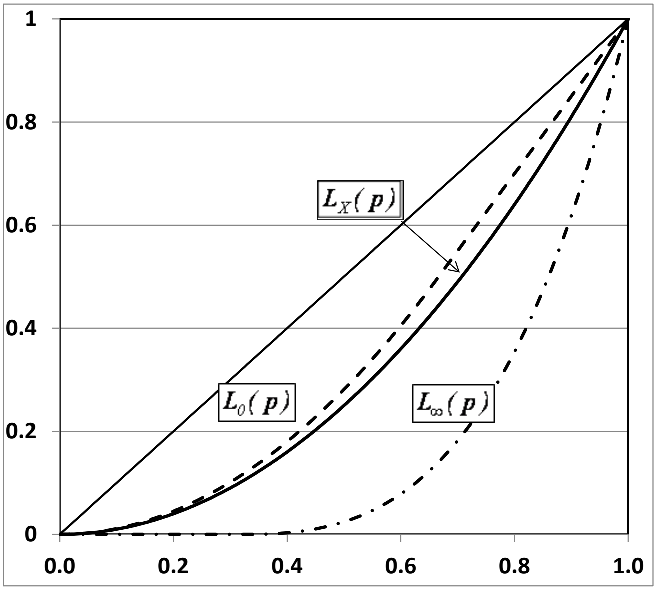


***Figure 3.1.2*** *The region between the extreme Lorenz curves is the region of attainable Lorenz curves (Fellman, 2001, 2014).*

We can prove (Fellman, 2001).

**Theorem 3.1.2.** The Lorenz curve $L_\infty(p)$ is inferior to all Lorenz curves corresponding to the class **U**.

**Proof.** Consider an arbitrary policy $u(x)$ in the class ***U***. For $x < c_\infty$, we get $u(x) \geq u_\infty(x)$. As a consequence of the condition $u'(x) \leq 1$ the curve $u(x)$ crosses $u_\infty(x)$ in only one point (say) $c_0 > c_\infty$ and for large $x$ values $u(x) \leq u_\infty(x)$. Hence, $u(x) \geq u_\infty(x)$ for $x < c_0$ and $u(x) \leq u_\infty(x)$ for $x \geq c_0$. Furthermore,

$$\int_0^\infty u_\infty(x) f_X(x)dx = \int_0^\infty u(x) f_X(x)dx = \mu_X - \tau .$$

The difference

$$D(p)=\int_0^{x_p}\left(u(x)-u_\infty(x)\right)f_X(x)dx$$

increases monotonically from 0 to a maximum $\int_0^{c_0}\left(u(x)-u_\infty(x)\right)f_X(x)dx$ for $x_p=c_0$ whereupon it decreases monotonically to 0 for $x_p\to\infty$. This behaviour proves the theorem.

The extreme Lorenz curves $L_0(p)$ and $L_\infty(p)$ define a closed region of attainable Lorenz curves (c.f. Figure 3.1.2).

Now we evaluate the corresponding Gini coefficient $G_\infty$. Consider the function

$$L_m(p)=\frac{\mu_X}{\mu_X-\tau}\left(L_X(p)-L_X(r_\infty)\right)-\frac{c_\infty}{\mu_X-\tau}\left(p-r_\infty\right) \tag{3.1.19}$$

For $p<r_\infty$, $L_m(p)<0$ and for $p\geq r_\infty$ $L_m(p)=L_\infty(p)$. Hence $L_\infty(p)\geq L_m(p)$ for all $p\in[0,1]$. If we use (3.1.17), we get

$$=1-\frac{\mu_X}{\mu_X-\tau}(1-G_X)+2L_X(r_\infty)-\frac{c_\infty}{\mu_X-\tau}(1-r_\infty)^2=$$

$$<G_X+\frac{\tau(1+G_X)}{\mu_X-\tau}-\frac{c_\infty}{\mu_X-\tau}\leq G_X+\frac{\tau(1+G_X)}{\mu_X-\tau}.$$

In fact, this upper bound $G_X+\frac{\tau(1+G_X)}{\mu_X-\tau}$ is the same as the bound given in Fellman (1995). There the bound was stricter, since it was obtained without the derivative restriction in (3.1.1).

As a consequence of the formula (3.1.14) and Theorem 3.1.2, the Gini coefficient $G_X + \frac{\tau(1+G_X)}{\mu_X - \tau}$ is the maximum and $G_X - \frac{\tau}{\mu_X - \tau}(1-G_X)$ is the minimum of the Gini coefficients for the class ***U***. Hence we obtain for every policy $u(x)$ the inequalities

$$G_X - \frac{\tau(1-G_X)}{\mu_X - \tau} \le G_u \le G_X + \frac{\tau(1+G_X)}{\mu_X - \tau}. \tag{3.1.20}$$

For the generalized Gini coefficient,

$$G(\nu) = 1 - \nu(1-\nu)\int_0^1 (1-p)^{\nu-2} L(p)dp$$

proposed by Yitzhaki (1983), Fellman (2001) obtained a similar formula

$$G_X(\nu) - \frac{\tau(1-G_X(\nu))}{\mu_X - \tau} \le G_u(\nu) \le G_X(\nu) + \frac{\tau(\nu-1+G_X(\nu))}{\mu_X - \tau}. \tag{3.1.21}$$

For $\nu = 2$ the formula (3.1.21) is identical with (3.1.20).

Consider the welfare index $W = \mu(1-G)$ developed by Sen (1973) and later discussed by Lambert (2001, Chapter 5). For this index, Fellman (2001) obtained the simple inequality formula

$$W_X - 2\tau \le W_u \le W_X. \tag{3.1.22}$$

From the deduction of the bounds in (3.1.20), (3.1.21) and (3.1.22), it follows that the formulae hold for arbitrary pre-tax income distributions. For a specific pre-tax income distribution, these bounds can be sharpened. This can be explained in the following way. Let us consider the lower bound in (3.1.20). For all pre-tax income distributions, the Lorenz curve $L_{u_0}(p)$ has a linear part, which starts from $p = p_0$ and which corresponds to the tax-paying part of the population (c.f. formula (3.1.8) and Figure 3.1.2). The accuracy of the lower bound depends on this linear part. The value of $1 - p_0$ indicates the proportion of taxpayers in the population and the accuracy of the bound increases as $1 - p_0$ decreases. Hence, the lower bound is accurate when there are very few but very high-income taxpayers.

Now we consider the upper bound in (3.1.20). The Lorenz curve $L_\infty(p) \equiv 0$ for $p < r_\infty$ and this part of the Lorenz curve influences the accuracy (c.f. formula (3.1.18) and Figure 3.1.2). For small values of $c_\infty$ and $r_\infty$ we obtain good accuracy. This is the case when the tax-paying ability of the low-income individuals is good, i.e. they are not extremely poor.

The strength of the obtained bounds is that they are independent of the distribution $F_X(x)$ and depend only on the basic quantities $G_X$, $\mu_X$ and $\rho$. In addition, the formulae obtained, are simple functions of these quantities. Furthermore, we observe that if $\tau \to 0$ then both the upper and lower bounds in (3.1.19), (3.1.20) and (3.1.21) converge towards $G_X$ $G_X(\nu)$ and $W_X$, respectively, indicating that the approximations presented have not introduced any "bias".

We have observed that ***U*** contains policies that increase and decrease inequality. Therefore, the intervals given for the indices are wide and the obtained bounds cannot be used as approximations of the indices of a specific policy in ***U***. The central role of these intervals is that they define limits for attainable index values and consequently give indications of the redistributive power of the class ***U***.

Let $G_0$ be the Gini coefficient of the policy (3.1.8) and $G_\infty$ be the Gini coefficient of the policy (3.1.15). Obviously, $\min_{\boldsymbol{U}} G = G_0 < G_\infty = \max_{\boldsymbol{U}} G$. Now we prove that the set of Gini coefficients which corresponds to the class (3.1.1) is compact, that is:

**Theorem 3.1.3.** There is a member of the class **U** with a prescribed Gini coefficient $\tilde{G} \in [G_0, G_\infty]$.

**Proof.** Let the prescribed Gini coefficient be $\tilde{G} \in [G_0, G_\infty]$. Construct a member of the class U as a linear combination of (3.1.8) and (3.1.15). We get $\tilde{G} = \theta G_0 + (1-\theta) G_\infty$ and the prescribed value of the Gini coefficient is obtained for

$$\tilde{\theta} = \frac{\tilde{G} - G_0}{G_\infty - G_0}. \tag{3.1.23}$$

**Remark.** Theorem 3.1.3 says that there exists at least one member of the class ***U*** that results in a post-tax income distribution with a prescribed Gini coefficient within the closed interval $[G_0, G_\infty]$. In general, this policy is not unique, but the extreme coefficients $G_0$ and $G_\infty$ are attainable only by the extreme policies.

One can also prove the analogous theorem:

**Theorem 3.1.4.** There is a member of the class **U** whose Lorenz curve satisfies the condition, $L_u(\tilde{p}) = \tilde{l}$ where $\tilde{l} \in [L_\infty(\tilde{p}), L_0(\tilde{p})]$.

**Proof.** The solution can be constructed by a linear combination of the policies (3.1.8) and (3.1.15). The prescribed condition is obtained for $\tilde{l} = \theta L_\infty(\tilde{p}) + (1-\theta) L_0(\tilde{p})$ and

$$\tilde{\theta} = \frac{\tilde{l} - L_0(\tilde{p})}{L_\infty(\tilde{p}) - L_0(\tilde{p})}. \tag{3.1.24}$$

Every point within the closed region, limited by the Lorenz curves $L_0(p)$ and $L_\infty(p)$, is attainable by a Lorenz curve corresponding to a member of the class ***U***. This means that there exists a policy that gives a post-tax income distribution such that the lowest proportion $\tilde{p}$ of income receivers receives exactly the proportion $\tilde{l}$ of the total amount of post-tax income. Within the class ***U***, the solution is not necessarily unique.

Consider the Lorenz curve $L_X(p)$ and the Lorenz curve $L_u(p)$, for an arbitrary member of the class ***U***. According to the general theory, we have $L'_X(p) = \frac{x_p}{\mu_X}$ and $L'_u(p) = \frac{y_p}{\mu_X - \tau}$. Now, $y_p = u(x_p)$ and, hence, $y_p \le x_p$ and we obtain

$$\frac{L'_u(p)}{L'_X(p)} \le \frac{\mu_X}{\mu_X - \tau}. \tag{3.1.25}$$

This is a necessary restriction on feasible Lorenz curves for members of the class ***U***. In general, there may be Lorenz curves between the extreme ones that do not correspond to policies in the class ***U***. The inequality (3.1.25) indicates that the Lorenz curve for the transformed variables cannot differ markedly from the Lorenz curve of *X*. This is especially notable for small values of $\tau \left(\tau / \mu_X\right)$. For the extreme policies (3.1.8) and (3.1.15) equality in (3.1.25) is obtained for the sub intervals $(0 \le p \le p_0)$ and $(r_\infty \le p \le 1)$, respectively. These properties stress the optimality of the extreme policies.

## 3.2 Attainable Lorenz Curves

In this Section we present necessary and sufficient conditions under which a given Lorenz curve can be obtained by a member of the class ***U***. These conditions are related to stochastic dominance of first order. Maasoumi and Heshmati (2000) presented stochastic dominance of first, second and third order and how they can be defined by alternative equivalent conditions.

Let $V$ and $W$ be non-negative stochastic variables having the distributions $F_V(v)$ and $F_W(w)$, the means $\mu_V$ and $\mu_W$ and the Lorenz curves $L_V(p)$ and $L_W(p)$, respectively. Using our notations the Maasoumi and Heshmati definition of stochastic dominance of first order is:

**Definition 3.2.1**. *The variable V First Order Stochastic Dominates W if and only if any one of the following equivalent conditions holds:*

*i.* $E[g(V)] \ge E[g(W)]$ *for all increasing functions g.*

*ii.* $F_V(v) \le F_W(v)$ *for all v.*

*iii.* $v_p \ge w_p$ *for all* $0 \le p \le 1$.

In this study of income distributions we restrict our investigations on non-negative continuous stochastic variables. For these the Lorenz curves are differentiable and we can prove the following lemma.

**Lemma 3.2.1.** *Let V and W be continuous non-negative stochastic variables having the distributions* $F_V(v)$ *and* $F_W(w)$, *the means* $\mu_V$ *and* $\mu_W$ *and the Lorenz curves* $L_V(p)$ *and* $L_W(p)$, *respectively, then the conditions:*

*i. V first order stochastic dominates W.*

*ii.* $F_V(v) \le F_W(v)$ *for all v.*

*iii.* $v_p \ge w_p$ *for all p* $(0 < p < 1)$.

*iv.* $\frac{L'_W(p)}{L'_V(p)} \le \frac{\mu_V}{\mu_W}$ *for all p* $(0 < p < 1)$.

*are equivalent.*

**Proof.** The equivalence between (i), (ii) and (iii) is given in Definition 3.2.1. Now, we only have to prove the equivalence between (iv) and (iii) (say). The connection between (iii) and (iv) are the formulae

$$L'_V(p) = \frac{v_p}{\mu_V} \text{ and } L'_W(p) = \frac{w_p}{\mu_W}$$

a) Assume that (iii) holds

Now,

$$1 \ge \frac{w_p}{v_p} = \frac{\mu_W L'_W(p)}{\mu_V L'_V(p)},$$

$$\frac{L'_W(p)}{L'_V(p)} \le \frac{\mu_V}{\mu_W}$$

and (iv) is obtained.

b) Assume that (iv) holds. Now

$$\frac{\mu_V}{\mu_W} \geq \frac{L'_W(p)}{L'_V(p)} = \left(\frac{w_p}{\mu_W}\right)\left(\frac{v_p}{\mu_V}\right)^{-1},$$

$$1 \geq \frac{w_p}{v_p},\ v_p \geq w_p$$

and the proof is completed.

**Remark.** The condition (iv) in Lemma 3.2.1, being equivalent with (i), (ii) and (iii), indicates that we have obtained a new criterion for stochastic dominance of first order between two non-negative stochastic variables.

In Section 3.1 formula (3.1.25) we have noted that stochastic dominance of first order is a necessary condition that the transformed distribution is a post-tax income distribution corresponding to a policy of the class ***U***. In the following we obtain sufficient conditions.

At first we consider the class

$$\mathbf{U^*}: \begin{cases} u(x) \leq x \\ E(u(X)) = \mu_X - \tau \end{cases} \tag{3.2.1}$$

This class, presented in Fellman (1995) and in Fellman et al. (1996, 1999), is defined as the initial class ***U*** without the restriction

$$u'(x) \leq 1 \tag{3.2.2}$$

and consequently, $\boldsymbol{U} \subseteq \boldsymbol{U^*}$. Now we prove

**Theorem 3.2.1.** (Fellman, 2002, 2014) Consider a differentiable Lorenz curve $\overline{L}(p)$ and a stochastic variable Y with the corresponding distribution $\overline{F}_Y(y)$ with the mean $(\mu_X - \tau)$. Then the necessary and sufficient conditions that the Lorenz curve $\bar{L}(p)$ is an attainable Lorenz curve of a member of **U***, $\bar{F}_Y(y)$ being the corresponding distribution and $\bar{u}(x) = (\mu_X - \tau)\bar{L}'(F_X(x))$ being the corresponding transformation, is that one of the following equivalent conditions holds:

i. X first order stochastic dominates Y.

ii. $F_X(x) \le \bar{F}_Y(x)$ for all x.

iii. $y_p \le x_p$ for all p ($0 < p < 1$) or.

iv. $\frac{\bar{L}'(p)}{L'_X(p)} \le \frac{\mu_X}{\mu_X - \tau}$ for all p ($0 < p < 1$).

**Proof.** Assume that the presumptive post tax income distribution is $\bar{F}_Y(y)$ ($\bar{f}_Y(y)$) with the mean $\mu_X - \tau$. We introduce the quantiles $x_p$ and $y_p$, where $F_X(x_p) = p$ and $\bar{F}_Y(y_p) = p$. These quantiles can also be defined as $x_p = F_X^{-1}(p)$ and $y_p = \bar{F}_Y^{-1}(p)$. In Section 3.1 we noted that

$$y_p \le x_p \text{ for all p } (0 < p < 1) \qquad (3.2.3)$$

and this condition still holds for the class ***U****. Consequently it is a necessary condition for $\bar{F}_Y(y)$ to be an attainable post-tax income distribution. From (3.2.3) it follows that

$$F_X(x_p) = p = \bar{F}_Y(y_p) \le \bar{F}_Y(x_p) \text{ for all p } (0 < p < 1).$$

The condition

$$F_X(x) \le \bar{F}_Y(x) \text{ for all x} \qquad (3.2.4)$$

being equivalent with (3.2.3) is also a necessary condition that the post-tax income distribution $\overline{F}_Y(y)$ corresponds to a tax policy belonging to ***U****. From formula (3.2.4) we obtain

$$\overline{F}_Y^{-1}(F_X(x)) \leq \overline{F}_Y^{-1}(\overline{F}_Y(x)) = x \text{ for all x.} \tag{3.2.5}$$

In the following we prove that the condition that the distribution $\overline{F}_Y(y)$ satisfies (3.2.5) is sufficient, that is, $\overline{F}_Y(y)$ is a post-tax income distribution for a member of the class ***U****. Consequently, the condition (3.2.4), being equivalent, is also sufficient. Consider a distribution $\overline{F}_Y(y)$ with mean $\mu_X - \tau$ satisfying (3.2.3). According to the definition of a distribution function we have

$$P(Y \leq y) = \overline{F}_Y(y)\,. \tag{3.2.6}$$

The cumulative distribution function $\overline{F}_Y(y)$ is monotone increasing and

$$P(\overline{F}_Y(Y) \leq \overline{F}_Y(y)) = \overline{F}_Y(y)\,. \tag{3.2.7}$$

If $Z = \overline{F}_Y(Y)$ and $z = \overline{F}_Y(y)$, then $Y = \overline{F}_Y^{-1}(Z)$, $y = \overline{F}_Y^{-1}(z)$ and

$$P(Z \leq z) = z\,. \tag{3.2.8}$$

Consider the initial distribution $F_X(x)$. Then

$$z = P(Z \leq z) = P(F_X^{-1}(Z) \leq F_X^{-1}(z))\,. \tag{3.2.9}$$

Let $X = F_X^{-1}(Z)$ and $x = F_X^{-1}(z)$ then $Z = F_X(X)$ and $z = F_X(x)$.

Now,

$$y = \bar{F}_Y^{-1}(z) = \bar{F}_Y^{-1}(F_X(x)) = \bar{u}(x) \text{ (say).} \tag{3.2.10}$$

Hence $\bar{u}(x)$ is continuous and monotone increasing. In addition, from (3.2.5) follows that $\bar{u}(x)$ satisfies the condition

$$\bar{u}(x) \le x \tag{3.2.11}$$

and $\bar{u}(x)$ belongs to ***U****** and the distribution $\bar{F}_Y(y)$ corresponds to a policy belonging to the class ***U****** and the sufficiency is obtained.

Let us now consider Lorenz curves. First we give the conditions that a specific Lorenz curve (and the corresponding distribution $\bar{F}_Y(y)$ ) can be attained by a member of the class ***U******. Let us consider an arbitrary Lorenz curve $\bar{L}(p)$ with the conditions

i. $\bar{L}(p)$ has a continuous derivative of the first order ($\bar{L}'(p)$).

ii. $\lim_{p \to 1}(1-p)\bar{L}'(p) = 0$.

These conditions imply that the corresponding distribution $\bar{F}_Y(y) = M(\frac{y}{\mu})$, where $M(\cdot)$ is the inverse function to $\bar{L}'(p)$, is continuous and has a finite mean μ. When the Lorenz curve $\bar{L}(p)$ and the mean μ are given then the corresponding income distribution is unique (Fellman, 1976, 1980).

Consider a Lorenz curve $\bar{L}(p)$ and the corresponding distribution $\bar{F}_Y(y)$ with the mean $\mu_X - \tau$. We have

$$\bar{L}'(p) = \frac{y_p}{\mu_X - \tau}$$

and

$$y_p = \bar{F}_Y^{-1}(p) = (\mu_X - \tau)\bar{L}'(p) = (\mu_X - \tau)\bar{L}'(F_X(x_p)).$$

From these formulae it follows that $\bar{u}(x) = (\mu_X - \tau)\bar{L}'(F_X(x))$. Hence, the condition

$$\bar{u}(x) = (\mu_X - \tau)\bar{L}'(F_X(x)) \leq x \qquad (3.2.12)$$

is a necessary condition for attainability. On the other hand let us assume that (3.2.12) holds. Let $\bar{F}_Y(y)$ be the distribution, which corresponds to $\bar{L}(p)$ and has the mean $\mu_X - \tau$. Then

$$x_p \geq (\mu_X - \tau)\bar{L}'(p) = (\mu_X - \tau)\bar{L}'(\bar{F}_Y(y_p)) = y_p,$$

and

$$y_p \leq x_p \text{ for all p } (0 < p < 1). \qquad (3.2.13)$$

Consequently, the condition (3.2.12) is also sufficient and the theorem is proved.

Now we add the restriction $u'(x) \leq 1$ and consider the initial class ***U*** of policies. For this class the necessary and sufficient condition is given in

**Theorem 3.2.2.** Consider a twice differentiable Lorenz curve $\bar{L}(p)$ and a stochastic variable Y with the corresponding distribution $\bar{F}_Y(y)$ with the mean $(\mu_X - \tau)$ and define $\bar{u}(x) = \bar{F}_Y^{-1}(F_X(x))$. Then necessary and sufficient condition that the Lorenz curve $\bar{L}(p)$ is an attainable Lorenz curve of a member of **U**, $\bar{F}_Y(y)$ being the corresponding distribution and $\bar{u}(x)$ being the corresponding transformation, is that one of the following equivalent conditions holds:

i. $\dfrac{\bar{L}''(p)}{L_X''(p)} \le \dfrac{\mu_X}{\mu_X - \tau}$ for all p ($0 < p < 1$) or equivalently,

ii. $f_X(x) \le \bar{f}_Y(y)$ where $y = \bar{F}_Y^{-1}(F_X(x)) = \bar{u}(x)$.

**Proof.** If we assume that $\bar{L}(p)$ has the second derivative $\bar{L}''(p)$ we can add the restriction $\bar{u}'(x) \le 1$ into (3.2.1) in order to obtain the class 3.1.1. We have the derivatives of first order $\bar{L}'(p) = \dfrac{y_p}{\mu_X - \tau}$ and $L_X'(p) = \dfrac{x_p}{\mu_X}$. According to the formula (1.3.2) the derivatives of the second order are $\bar{L}''(p) = \dfrac{1}{(\mu_X - \tau)\bar{f}_Y(y_p)}$ and $L_X''(p) = \dfrac{1}{\mu_X f_X(x_p)}$.

Note that if we, according to (3.2.10), define $y = \bar{F}_Y^{-1}(F_X(x)) = \bar{u}(x)$ then we obtain

$$\bar{u}'(x) = \frac{f_X(x)}{\bar{f}_Y(\bar{F}_Y^{-1}(F_X(x))} \le 1. \tag{3.2.14}$$

and

$$f_X(x) \le \bar{f}_Y(y) \text{ where } y = \bar{F}_Y^{-1}(F_X(x)) = \bar{u}(x) \tag{3.2.15}$$

Hence, for every $p$ ($0 < p < 1$) we have

$$f_X(x_p) \le \bar{f}_Y(y_p). \tag{3.2.16}$$

Consequently, (3.2.16) can be written

$$\frac{\bar{L}''(p)}{L''_X(p)} \leq \frac{\mu_X}{\mu_X - \tau}. \tag{3.2.17}$$

This is a necessary condition that the transformation $y = \bar{F}_Y^{-1}(F_X(x)) = \bar{u}(x)$ in (3.2.14) belongs to the class ***U***. We can reverse the steps from (3.2.17) to (3.2.14) and consequently, (3.2.17) is also sufficient and the proof is completed.

If (3.2.17) is integrated we obtain

$$\mu_X L'_X(p) \geq (\mu_X - \tau)\bar{L}'(p), \tag{3.2.18}$$

or alternatively

$$\frac{\bar{L}'(p)}{L'_X(p)} \leq \frac{\mu_X}{(\mu_X - \tau)} \tag{3.2.19}$$

which is identical with the condition (3.1.25). The integration step from (3.2.17) to (3.2.19) is not reversible so the condition (3.2.19) is only necessary for the class ***U*** given in (3.1.1) but, as proved above, necessary and sufficient for the class *U** given in (3.2.1). This difference can be explained so that there can exist policies belonging to the class *U** but not belonging to ***U***. Explicitly, such policies do not satisfy the condition $\bar{u}'(x) \leq 1$.

The condition (iv) implies after integrations that

$$(\mu_X - \tau)\bar{L}(p) \leq \mu_X L_X(p). \tag{3.2.20}$$

indicating *Generalized Lorenz Dominance* (GLD). The integration step from $\bar{L}'(p) \leq \frac{\mu_X}{(\mu_X - \tau)} L'_X(p)$ given in (iv) in Theorem 3.2.1 to the condition (3.2.20) is not reversible. Consequently, GLD is only a necessary condition, or otherwise expressed, stochastic dominance implies GLD (cf. Lambert 2001 p. 49).

## 3.3 Classes of Non-differentiable Tax Policies

The transformed variable $Y = u(X)$ is the income after the taxation (Fellman, 2001, 2002; Fellman et al., 1996, 1999). In order to obtain a realistic class of policies we included in Fellman (2001, 2002) the additional restriction $u'(x) \le 1$. This condition indicates that the tax paid is an increasing function of the income *x*. In order to allow that the function $u(x)$ is not differentiable everywhere, we replace in this study the derivative restriction by the more general condition $\Delta u(x) \le \Delta x$ (Fellman, 2013). According to this restriction the function $u(x)$ is continuous and the tax is an increasing function of the income *x*. In fact, the increment in the tax is $\Delta x - \Delta u(x) \ge 0$. If $u'(x) \le 1$ holds then it follows that

$$\Delta u(x) = u(x + \Delta x) - u(x) = u'(\xi)\Delta x \le \Delta x ,$$

but the condition $\Delta u(x) \le \Delta x$ is more general and does not imply differentiability. We intend to show that the assumption $\Delta u(x) \le \Delta x$ is sufficient for the whole theory.

Now, the class of tax policies is

$$\mathbf{U:} \begin{cases} u(x) \le x \\ \Delta u(x) \le \Delta x \\ E(u(X)) = \mu_X - \tau \end{cases} . \qquad (3.3.1)$$

We consider the extreme policies

$$u_0(x) = \begin{cases} x & x \le a_0 \\ a_0 & x > a_0 \end{cases} \qquad (3.3.2)$$

and

$$u_\infty(x) = \begin{cases} 0 & x \le c_\infty \\ x - c_\infty & x > c_\infty \end{cases}. \qquad (3.3.3)$$

The function $u_0(x)$ in (3.3.2) is not differentiable in the point $a_0$ and $u_\infty(x)$ in (3.3.3) in the point $c_\infty$, but the condition $\Delta u(x) \le \Delta x$ holds for all $x$. Already in (3.1.12) we obtained that the Lorenz curve corresponding to (3.3.2) is

$$L_0(p) = \begin{cases} \dfrac{\mu_X}{\mu_X - \tau} L_X(p) & p \le p_0 \\ \dfrac{\mu_X}{\mu_X - \tau} L_X(p_0) + \dfrac{a_0}{\mu_X - \tau}(p - p_0) & p > p_0 \end{cases}, \qquad (3.3.4)$$

where $p_0 = F_X(a_0)$ and according to (3.1.18) the Lorenz curve corresponding to (3.3.3) is

$$L_\infty(p) = \begin{cases} 0 & p < r_\infty \\ \dfrac{\mu_X}{\mu_X - \tau}\left(L_X(p) - L_X(r_\infty)\right) - \dfrac{c_\infty(p - r_\infty)}{\mu_X - \tau} & p \ge r_\infty \end{cases}, \qquad (3.3.5)$$

where $p_\infty = F_X(c_\infty)$.

The policy (3.3.2) is optimal, that is, it Lorenz dominates all the policies in the class **U**, and the policy (3.3.3) is Lorenz dominated by all policies in **U** (Fellman, 2001, 2002).

In the following we show how the main result in Fellman (2002) can be obtained when we replace the restriction $\bar{u}?\, x) \le 1$ by the more general restriction $\Delta\bar{u}(x) \le \Delta x$. The function $\bar{u}(x)$ may be piecewise differentiable as the transformations (3.3.2) and (3.3.3). We consider post-tax income distributions with the mean $\mu_X - \tau$. Without the restriction $\Delta\bar{u}(x) \le \Delta x$, the necessary and sufficient condition that a given Lorenz curve $\overline{L}(p)$ ($\overline{F}_Y(y)$) corresponds to a member of the class **U** is that the initial distribution $F_X(x)$ stochastically dominates $\overline{F}_Y(y)$. The inclusion of the restriction $\Delta\overline{u}(x) \le \Delta x$ results that the stochastic dominance is only necessary, that is the transformed distribution $\overline{F}_Y(y)$ must satisfy additional conditions.

Assume a given differentiable Lorenz curve $\overline{L}(p)$ with a continuous derivative. These conditions can be assumed because the corresponding transformation $\overline{u}(x)$ has to be continuous satisfying the condition $\Delta\overline{u}(x) \le \Delta x$. Starting from $\overline{L}(p)$, the connection between $\overline{L}(p)$ and the post-tax distribution $\overline{F}_Y(y)$ with the mean $\mu_X - \tau$ is that $\overline{F}_Y(y) = M\left(\frac{y}{\mu_X - \tau}\right)$, where $M(\cdot)$ is the inverse function of $\overline{L}'(p)$. The corresponding transformation is $\overline{u}(x) = y = (\mu_X - \tau)\overline{L}'(F_X(x))$. The condition $\Delta\overline{u}(x) \le \Delta x$ can be written

$$\begin{aligned}\Delta\overline{u}(x) &= (\mu_X - \tau)\left(\overline{L}'(F_X(x+\Delta x)) - \overline{L}'(F_X(x))\right)\\ &= (\mu_X - \tau)\left(\overline{L}'(p+\Delta p) - \overline{L}'(p)\right)\end{aligned}$$

where $p = F_X(x)$ and $p + \Delta p = F_X(x + \Delta x)$. On the other hand, we can write

$$\Delta\overline{u}(x) = (\mu_X - \tau)\left(\overline{L}'(p+\Delta p) - \overline{L}'(p)\right) = \left(y_{p+\Delta p} - y_p\right),$$

where $y_p$ and $y_{p+\Delta p}$ are defined by $p = \overline{F}_Y(y_p)$, $p + \Delta p = \overline{F}_Y(y_{p+\Delta p})$.

If we assume that $\overline{u}(x)$ is piecewise differentiable, then $\overline{L}'(p)$ and $\overline{F}_Y(y)$ are piecewise differentiable.

If we assume that the density functions $f_X(x)$ and $\bar{f}_Y(y)$ exist, we obtain

$$\Delta p = F_X(x+\Delta x) - F_X(x) = f_X(\xi)\Delta x,$$

where $x < \xi < x + \Delta x$ and

$$\begin{aligned}\Delta p &= \bar{F}_Y(y_{p+\Delta p}) - \bar{F}_Y(y_p) = \bar{f}_Y(\eta)(y_{p+\Delta p} - y_p) \\ &= \bar{f}_Y(\eta)\big(\bar{u}(x_p + \Delta x) - \bar{u}(x_p)\big)\end{aligned}$$

where $\bar{f}_Y(y) = \bar{F}_Y'(y)$ and $y_p < \eta < y_{p+\Delta p}$.

Consequently,

$$p = \bar{F}_Y(y_p) = F_X(x_p)$$

and

$$y_p = \bar{F}_Y^{-1}\big(F_X(x)\big).$$

From $f_X(\xi)\Delta x = \Delta p = \bar{f}_Y(\eta)\Delta\bar{u}(x)$ and from the condition $\Delta\bar{u}(x) \le \Delta x$ it follows that

$$f_X(\xi)\Delta x = \bar{f}_Y(\eta)\Delta\bar{u}(x) \le \bar{f}_Y(\eta)\Delta x$$

and consequently, $\dfrac{f_X(\xi)}{\bar{f}_Y(\eta)} \le 1$. If we let $\Delta x \to 0$, then $\Delta p \to 0$, $\xi \to x$ and $\eta \to y_p$ and we obtain $\dfrac{f_X(x)}{\bar{f}_Y(y_p)} \le 1$. This condition can also be written $h(x)$ or $\dfrac{f_X(x)}{\bar{f}_Y(y)} \le 1$ when $h(x)$. Hence, all the results in Fellman (2002) still hold, but the proof had to be slightly modified.

## 3.4 Discussion

In this chapter we reconsidered the effect of variable transformations on the redistribution of income. The aim was to generalise the conditions considered in earlier papers. Particularly we were interested if we can drop the assumptions of continuity and differentiability of the transformations. The main result is that with a slight modification of the proof the additional condition $\frac{f_X(x)}{\bar{f}_Y(y)} \leq 1$ is obtained.

We have obtained that, if we demand sufficient and necessary conditions, theorems earlier obtained, still hold and the continuity assumption can be included in the general conditions. The main result is that continuity is a necessary condition if one pursues that the income inequality should remain or be reduced.

The study of the class of tax policies indicated that the differentiability, earlier assumed, can be dropped but if one wants to retain the realism of the class the transformations should still be continuous and satisfy the restriction $\Delta\bar{u}(x) \leq \Delta x$. The earlier results obtained and presented in Fellman (2001, 2002) still hold.

Empirical applications of the optimal policies of a class of tax policies and the class of transfer policies considered here have been discussed in Fellman et al. (1996, 1999). There we developed "optimal yardsticks" to gauge the effectiveness of given real tax and transfer policies in reducing inequality.

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