# Chapter 4

# Zweier I-Convergent Sequence Spaces Defined by Orlicz Function

"Mathematics is a free flow of thoughts and concepts which a mathematicians, in the same way as musician does with the tones of music and a poet with words, puts together into theorems and theories"- Orlicz.

## 4.1 Introduction

An *Orlicz function* is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

If convexity of $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a *Modulus function*, defined and discussed by Nakano [58], Ruckle [62-64].

An Orlicz function $M$ can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Lindenstrauss and Tzafriri [55] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\};$$

which is a Banach space with the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Remark . An Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

For more details on Orlicz sequence spaces we refer to [55], [21-28].

## 4.2 Main Results

In this chapter we introduce the following classes of sequence spaces:

$$\mathcal{Z}^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^/ - L|}{\rho}) = 0 \text{ for some L and } \rho > 0\},$$

$$\mathcal{Z}_0^I(M) = \{(x_k) \in \omega : I - \lim M(\frac{|x_k^/|}{\rho}) = 0 \text{ for some } \rho > 0\},$$

$$\mathcal{Z}_\infty^I(M) = \{(x_k) \in \omega : \sup_k M(\frac{|x_k^/|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

Also we denote by

$$m_{\mathcal{Z}}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}^I(M)$$

and

$$m_{\mathcal{Z}_0}^I(M) = \mathcal{Z}_\infty(M) \cap \mathcal{Z}_0^I(M).$$

Theorem 4.2.1. For any Orlicz function $M$, the classes of sequences $\mathcal{Z}^I(M), \mathcal{Z}_0^I(M), m_{\mathcal{Z}}^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I(M)$. The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I(M)$ and let $\alpha, \beta$ be scalars. Then there exists positive numbers $\rho_1$ and $\rho_2$ such that

$$I - \lim M(\frac{|x_k^/ - L_1|}{\rho_1}) = 0, \text{ for some } L_1 \in \mathbb{C};$$

$$I - \lim M(\frac{|y_k^/ - L_2|}{\rho_2}) = 0, \text{ for some } L_2 \in \mathbb{C}.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : M(\frac{|x_k^{/} - L_1|}{\rho_1}) > \frac{\epsilon}{2}\} \in I, \quad [4.1]$$

$$A_2 = \{k \in \mathbb{N} : M(\frac{|y_k^{/} - L_2|}{\rho_2}) > \frac{\epsilon}{2}\} \in I. \quad [4.2]$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $M$ is non-decreasing and convex function, we have

$$M(\frac{|(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\leq M(\frac{|\alpha||x_k^{/} - L_1|}{\rho_3}) + M(\frac{|\beta||y_k^{/} - L_2|}{\rho_3})$$

$$\leq M(\frac{|x_k^{/} - L_1|}{\rho_1}) + M(\frac{|y_k^{/} - L_2|}{\rho_2}).$$

Now, by [4.1] and [4.2],

$$\{k \in \mathbb{N} : M(\frac{|(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore

$$(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(M).$$

Hence $\mathcal{Z}^I(M)$ is a linear space.

Theorem 4.2.2. The spaces $m_{\mathcal{Z}}^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are Banach spaces normed by

$$||x_k|| = \inf\{\rho > 0 : \sup_k M(\frac{|x_k|}{\rho}) \leq 1\}.$$

Proof. Proof of this result is easy in view of the existing techniques and therefore is omitted.

Theorem 4.2.3. Let $M_1$ and $M_2$ be Orlicz functions that satisfy the $\triangle_2$-condition. Then

[a] $X(M_2) \subseteq X(M_1.M_2)$;

[b] $X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Proof. [a] Let $(x_k) \in \mathcal{Z}_0^I(M_2)$. Then there exists $\rho > 0$ such that

$$I - \lim_k M_2(\frac{|x_k^/|}{\rho}) = 0. \qquad [4.3]$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq t \leq \delta$. Write

$$y_k = M_2(\frac{|x_k^/|}{\rho}),$$

and consider

$$\lim_{0 \leq y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) = \lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_1(y_k).$$

We have

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} M_1(y_k) \leq M_1(2). \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k). \qquad [4.4]$$

For $(y_k) > \delta$, we have

$$(y_k) < (\frac{y_k}{\delta}) < 1 + (\frac{y_k}{\delta}).$$

Since $M_1$ is non-decreasing and convex, it follows that

$$M_1(y_k) < M_1(1 + (\frac{y_k}{\delta})) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_k}{\delta}).$$

Since $M_1$ satisfies the $\triangle_2$-condition, we have

$$M_1(y_k) < \frac{1}{2}K(\frac{y_k}{\delta})M_1(2) + \frac{1}{2}K(\frac{y_k}{\delta})M_1(2) = K(\frac{y_k}{\delta})M_1(2).$$

Hence

$$\lim_{y_k>\delta,k\in\mathbb{N}} M_1(y_k) \leq max(1, K\delta^{-1}M_1(2)) \lim_{y_k>\delta,k\in\mathbb{N}}(y_k). \qquad [4.5]$$

From [4.3], [4.4] and [4.5], we have $(x_k) \in \mathcal{Z}_0^I(M_1.M_2)$. Thus

$$\mathcal{Z}_0^I(M_2) \subseteq \mathcal{Z}_0^I(M_1.M_2).$$

The other cases can be proved similarly.

[b] Let

$$(x_k) \in \mathcal{Z}_0^I(M_1) \cap \mathcal{Z}_0^I(M_2).$$

Then there exists $\rho > 0$ such that

$$I - \lim_k M_1(\frac{|x_k^{/}|}{\rho}) = 0$$

and

$$I - \lim_k M_2(\frac{|x_k^{/}|}{\rho}) = 0.$$

The rest of the proof follows from the following equality

$$\lim_{k\in\mathbb{N}}(M_1 + M_2)(\frac{|x_k^{/}|}{\rho}) = \lim_{k\in\mathbb{N}} M_1(\frac{|x_k^{/}|}{\rho}) + \lim_{k\in\mathbb{N}} M_2(\frac{|x_k^{/}|}{\rho}).$$

Therem 4.2.4. The spaces $\mathcal{Z}_0^I(M)$ and $m_{\mathcal{Z}_0}^I(M)$ are solid and monotone.

Proof. We shall prove the result for $\mathcal{Z}_0^I(M)$. For $m_{\mathcal{Z}_0}^I(M)$ the result can be proved similarly. Let $(x_k) \in \mathcal{Z}_0^I(M)$. Then there exists $\rho > 0$ such that

$$I - \lim_k M(\frac{|x_k^{/}|}{\rho}) = 0. \qquad [4.6]$$

Let $(\alpha_k)$ be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [4.6] and the following inequality

$$M(\frac{|\alpha_k x_k^{/}|}{\rho}) \leq |\alpha_k| M(\frac{|x_k^{/}|}{\rho}) \leq M(\frac{|x_k^{/}|}{\rho}) \text{ for all } k \in \mathbb{N}.$$

By Lemma 4.1.1, a sequence space E is solid implies that E is monotone. We have the space $\mathcal{Z}_0^I(M)$ is monotone.

Theorem 4.2.5. The spaces $\mathcal{Z}^I(M)$ and $m_{\mathcal{Z}}^I(M)$ are neither solid nor monotone in general.

Proof. Here we give a counter example.

Let $I = I_\delta$ and $M(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(M)$ of $X(M)$ defined as follows, let $(x_k) \in X(M)$ and let $(y_k) \in X_K(M)$ be such that

$$y_k = \begin{cases} x_k, & \text{if k is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence $x_k$ defined by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I(M)$ but its $K$-stepspace preimage does not belong to $\mathcal{Z}^I(M)$. Thus $\mathcal{Z}^I(M)$ is not monotone. Hence $\mathcal{Z}^I(M)$ is not solid.

Theorem 4.2.6. The spaces $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$ and $M(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence $(x_k)$ and $(y_k)$ defined by

$$x_k = \frac{1}{k} \quad and \;\; y_k = k \;\text{ for all } k \in \text{N}.$$

Then $(x_k) \in \mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$, but $(y_k) \notin \mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$. Hence the spaces $\mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$ are not convergence free.

Theorem 4.2.7. The spaces $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I(M)$ is a sequence algebra. For the space

$\mathcal{Z}^I(M)$, the result can be proved similarly. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(M)$. Then

$$I - \lim M(\frac{|x_k^/|}{\rho_1}) = 0 \quad \text{for some} \quad \rho_1 > 0$$

and

$$I - \lim M(\frac{|y_k^/|}{\rho_2}) = 0 \quad \text{for some} \quad \rho_2 > 0.$$

Let $\rho = \rho_1.\rho_2 > 0$. Then we can show that

$$I - \lim M(\frac{|(x_k^/.y_k^/)|}{\rho}) = 0.$$

Thus

$$(x_k.y_k) \in \mathcal{Z}_0^I(M).$$

Hence $\mathcal{Z}_0^I(M)$ is a sequence algebra.

Theorem 4.2.8. Let $M$ be an Orlicz function. Then the inclusions $\mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M) \subset \mathcal{Z}_\infty^I(M)$ hold.

Proof. Let $(x_k) \in \mathcal{Z}^I(M)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$I - \lim M(\frac{|x_k^/ - L|}{\rho}) = 0.$$

We have

$$M(\frac{|x_k^/|}{2\rho}) \leq \frac{1}{2}M(\frac{|x_k^/ - L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho}).$$

Taking supremum over k both sides we get

$$(x_k) \in \mathcal{Z}_\infty^I(M).$$

The inclusion

$$\mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M)$$

is obvious.

Theorem 4.2.9. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(M)$ and $\mathcal{Z}_0^I(M)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $M(x) = x$ for all $x \in [0, \infty)$. If

$$x_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in \mathcal{Z}_0^I(M) \subset \mathcal{Z}^I(M)$, by lemma 3.1.8. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \text{otherwise.} \end{cases}$$

is a permutation on $\mathbb{N}$, but $(x_{\pi(k)}) \notin \mathcal{Z}^I(M)$ and $(x_{\pi(k)}) \notin \mathcal{Z}_0^I(M)$. Hence $\mathcal{Z}_0^I(M)$ and $\mathcal{Z}^I(M)$are not symmetric.