Part I

Axiomatic K-theory

Throughout *Part I* we endow $\{0,1\}$ with the structure of a group by identifying it with $\mathbf{Z}_2$ and take $i \in \{0,1\}$.

Chapter 1

The Axiomatic Theory

1.1 E-C*-algebras

DEFINITION 1.1.1 *In this book we call E***-C*-algebra** *a C*-algebra F endowed with a bilinear map (exterior multiplication)*

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

such that for all $\alpha, \beta \in E$ and $x, y \in F$,

$$(\alpha+\beta)x = \alpha x + \beta y, \quad (\alpha\beta)x = \alpha(\beta x), \quad (\alpha x)^* = \alpha^* x^*, \quad \|\alpha x\| \leq \|\alpha\| \, \|x\|,$$

$$\alpha(x+y) = \alpha x + \alpha y, \qquad \alpha(xy) = (\alpha x)y = x(\alpha y), \qquad 1_E x = x.$$

*An E***-C*-subalgebra (***E***-ideal)** *of F is a C*-subalgebra (a closed ideal) G of F such that*

$$(\alpha, x) \in E \times G \Longrightarrow \alpha x \in G.$$

If F, G are E-C-algebras then a C*-homomorphism $\varphi : F \longrightarrow G$ is called E***-linear** *or an E***-C*-homomorphism** *if for all $(\alpha, x) \in E \times F$, $\varphi(\alpha x) = \alpha \varphi x$. A bijective E-C*-homomorphism is called E***-C*-isomorphism**. *We denote by 0 the E-C*-algebra having a unique element. We denote by $\mathfrak{M}_E$ the category of E-C*-algebras for which the morphisms are the E-linear C*-homomorphisms. In particular $\mathfrak{M}_{\mathbb{C}}$ is the category of all C*-algebras.*

If G is an E-ideal of the E-C-algebra F then the C*-algebra F/G has a natural structure of an E-C*-algebra and*

$$0 \longrightarrow G \overset{\varphi}{\longrightarrow} F \overset{\psi}{\longrightarrow} F/G \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$, where φ denotes the inclusion map and ψ the quotient map. Conversely, if

$$0 \longrightarrow F \overset{\varphi}{\longrightarrow} G \overset{\psi}{\longrightarrow} H \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ then F is an E-ideal of G and $H \approx G/F$.

DEFINITION 1.1.2 *If $(F_j)_{j \in J}$ is a finite family of E-C*-algebras then we denote by $\prod_{j \in J} F_j$ the E-C*-algebra obtained by endowing the corresponding C*-algebra $\prod_{j \in J} F_j$ with the bilinear map*

$$E \times \prod_{j \in J} F_j \longrightarrow \prod_{j \in J} F_j, \quad (\alpha, (x_j)_{j \in J}) \longmapsto (\alpha x_j)_{j \in J}.$$

PROPOSITION 1.1.3 *Every C*-algebra can be endowed with the structure of an E-C*-algebra.*

Let F be a C*-algebra. Let Ω be the spectrum of E and $\omega \in \Omega$ and put

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha(\omega)x\,.$$

It is easy to see that F endowed with this exterior multiplication is an E-C*-algebra. ■

EXAMPLE 1.1.4 *Let Ω be a finite set and $E := \mathscr{C}(\Omega, \mathbb{C})$.*

a) *Let $(F_\omega)_{\omega\in\Omega}$ be a finite family of C*-algebras and $F := \prod_{\omega\in\Omega} F_\omega$. If we put for all $(\alpha, x) \in E \times F$,*

$$\alpha x : \Omega \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.*

b) *Let F be an E-C*-algebra and for every $\omega \in \Omega$ put*

$$e_\omega : \Omega \longrightarrow \mathbb{C}\,, \quad \omega' \longmapsto \begin{cases} 1 & \text{if} \quad \omega' = \omega \\ 0 & \text{if} \quad \omega' \neq \omega \end{cases}\,,$$

$$F_\omega := \{\, e_\omega x \mid x \in F \,\}\,.$$

Then F_ω is a C-algebra for all $\omega \in \Omega$ and $F \approx \prod_{\omega\in\Omega} F_\omega$, with the meaning of a).* ■

EXAMPLE 1.1.5 *Let Ω be a discrete locally compact space, Ω^* a compactification of Ω, $E := \mathscr{C}(\Omega^*, \mathbb{C})$, $(F_\omega)_{\omega\in\Omega}$ a family of C*-algebras, and $F := \prod_{\omega\in\Omega} F_\omega$ $\left(\text{resp. } F := \left\{ x \in \prod_{\omega\in\Omega} F_\omega \;\middle|\; \lim_{\omega\to\infty} \|x_\omega\| = 0 \right\}\right)$. If we put for all $(\alpha, x) \in E \times F$*

$$\alpha x : \Omega \longrightarrow F\,, \quad \omega \longmapsto \alpha(\omega)x_\omega$$

then $\alpha x \in F$ for all $(\alpha, x) \in E \times F$ and F endowed with the exterior multiplication

$$E \times F \longrightarrow F, \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.* ■

1.2 The Axioms

DEFINITION 1.2.1 *We denote by K_0 and K_1 two covariant functors from the category $\mathfrak{M}_E$ to the category of additive groups. We denote by 0 the group which has a unique element and call* **K-null** *an E-C^*-algebra F for which $K_i(F) = 0$. Let $F \stackrel{\varphi}{\to} G$ be a morphism in $\mathfrak{M}_E$. We say that φ is* **K-null** *if $K_i(\varphi) = 0$. We say that φ* **factorizes through null** *if there are morphisms $F \stackrel{\varphi'}{\to} H$ and $H \stackrel{\varphi''}{\to} G$ in $\mathfrak{M}_E$ such that $\varphi = \varphi'' \circ \varphi'$ and such that H is K-null.*

We have $K_i(id_F) = id_{K_i(F)}$ for every E-C*-algebra F. Every morphism which factorizes through null is K-null.

AXIOM 1.2.2 (Null-axiom) *$K_i(0) = 0$.*

AXIOM 1.2.3 (Split exact axiom) *If*

$$0 \longrightarrow F \stackrel{\varphi}{\longrightarrow} G \underset{\stackrel{\lambda}{\longleftarrow}}{\stackrel{\psi}{\longrightarrow}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ then

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \underset{\xleftarrow{K_i(\lambda)}}{\xrightarrow{K_i(\psi)}} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups.

It follows that the map

$$K_i(F) \times K_i(H) \longrightarrow K_i(G), \quad (a,b) \longmapsto K_i(\varphi)\,a + K_i(\lambda)\,b$$

is a group isomorphism.

DEFINITION 1.2.4 *Let $\varphi, \psi : F \longrightarrow G$ be morphisms in $\mathfrak{M}_E$. We say that φ and ψ are* **homotopic** *if there is a path*

$$\phi_s : F \longrightarrow G, \qquad s \in [0,1]$$

of morphisms in $\mathfrak{M}_E$ such that $\phi_0 = \varphi$, $\phi_1 = \psi$, and the map

$$[0,1] \longrightarrow G, \quad s \longmapsto \phi_s x$$

is continuous for every $x \in F$.

We say that a pair $F \xrightarrow{\varphi} G$, $G \xrightarrow{\psi} F$ of morphisms in $\mathfrak{M}_E$ is a **homotopy** *if $\psi \circ \varphi$ is homotopic to id_F and $\varphi \circ \psi$ is homotopic to id_G. In this case we say that F and G are* **homotopic**. *F is called* **null-homotopic** *if it is homotopic to the E-C*-algebra* 0.

AXIOM 1.2.5 (Homotopy axiom) *If $\varphi, \psi : F \longrightarrow G$ are homotopic morphisms in $\mathfrak{M}_E$ then $K_i(\varphi) = K_i(\psi)$.*

DEFINITION 1.2.6 *We associate to every exact sequence*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

in $\mathfrak{M}_E$ two group homomorphisms (called **index maps)**

$$\delta_i : K_i(H) \longrightarrow K_{i+1}(F) \ .$$

AXIOM 1.2.7 (Six-term axiom) *For every exact sequence in $\mathfrak{M}_E$*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

the six-term sequence

$$\begin{array}{ccccc} K_0(F) & \xrightarrow{K_0(\varphi)} & K_0(G) & \xrightarrow{K_0(\psi)} & K_0(H) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(H) & \xleftarrow[K_1(\psi)]{} & K_1(G) & \xleftarrow[K_1(\varphi)]{} & K_1(F) \end{array}$$

is exact.

AXIOM 1.2.8 (Commutativity of the index maps) *If the diagram in $\mathfrak{M}_E$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & H & \longrightarrow & 0 \\ & & \phi_1 \downarrow & & \phi_2 \downarrow & & \downarrow \phi_3 & & \\ 0 & \longrightarrow & F' & \xrightarrow[\varphi']{} & G' & \xrightarrow[\psi']{} & H' & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows then the diagram

$$\begin{array}{ccc} K_i(H) & \xrightarrow{\delta_i} & K_{i+1}(F) \\ {\scriptstyle K_i(\phi_3)}\downarrow & & \downarrow{\scriptstyle K_{i+1}(\phi_1)} \\ K_i(H') & \xrightarrow[\delta'_1]{} & K_{i+1}(F') \end{array}$$

is commutative, where δ_i *and* δ'_i *denote the index maps associated to the upper and the lower row of the above diagram, respectively.*

Remark. The above axioms are fulfilled if $K_i(F) = 0$ for all E-C*-algebras F .

1.3 Some Elementary Results

PROPOSITION 1.3.1 *If*

$$0 \longrightarrow F \xrightarrow{\varphi} G \underset{\underset{\lambda}{\longleftarrow}}{\overset{\psi}{\longrightarrow}} H \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ *then its index maps are* 0*.*

By the split exact axiom (Axiom 1.2.3),

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\varphi)} K_i(G) \underset{\underset{K_i(\lambda)}{\longleftarrow}}{\overset{K_i(\psi)}{\longrightarrow}} K_i(H) \longrightarrow 0$$

is a split exact sequence in the category of additive groups and the assertion follows from the six-term axiom (Axiom 1.2.7). ■

DEFINITION 1.3.2 *Let* $(F_j)_{j\in J}$ *be a finite family of E-C*-algebras,* $F := \prod_{j\in J} F_j$ *and for every* $j \in J$ *let* $\varphi_j : F_j \longrightarrow F$ *be the canonical inclusion and* $\psi_j : F \longrightarrow F_j$ *the canonical projection. We define*

$$\Phi_{(F_j)_{j\in J},i} : \prod_{j\in J} K_i(F_j) \longrightarrow K_i(F), \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j)\, a_j,$$

$$\Psi_{(F_j)_{j\in J},i} : K_i(F) \longrightarrow \prod_{j\in J} F_j, \quad a \longmapsto (K_i(\psi_j)\, a)_{j\in J}.$$

PROPOSITION 1.3.3 *If $(F_j)_{j\in J}$ is a finite family of E-C*-algebras then the map*

$$\Phi_{(F_j)_{j\in J},i} : \prod_{j\in J} K_i(F_j) \longrightarrow K_i\left(\prod_{j\in J} F_j\right)$$

is a group isomorphism and

$$\Psi_{(F_j)_{j\in J},i} : K_i\left(\prod_{j\in J} F_j\right) \longrightarrow \prod_{j\in J} K_i(F_j)$$

is its inverse.

If $J=\emptyset$ then the assertion follows from the null-axiom (Axiom 1.2.2). The assertion is trivial for $Card\, J = 1$. We prove the general case by induction with respect to $Card\, J$. Let $j_0 \in J$ and assume the assertion holds for $J' := J \setminus \{j_0\}$. We denote by

$$\varphi : F_{j_0} \longrightarrow \prod_{j\in J} F_j\,, \qquad \lambda : \prod_{j\in J'} F_j \longrightarrow \prod_{j\in J} F_j$$

the canonical inclusion maps and by

$$\psi : \prod_{j\in J} F_j \longrightarrow \prod_{j\in J'} F_j$$

the canonical projection. Then

$$0 \longrightarrow F_{j_0} \xrightarrow{\varphi} \prod_{j\in J} F_j \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \prod_{j\in J'} F_j \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$. By the split exact axiom (Axiom 1.2.3) the map

$$\Psi_i : K_i(F_{j_0}) \times K_i\left(\prod_{j\in J'} F_j\right) \longrightarrow K_i\left(\prod_{j\in J} F_j\right), \quad (a,b) \longmapsto K_i(\varphi)\, a + K_i(\lambda)\, b$$

is a group isomorphism. Since

$$\Psi_i \circ \left(id_{K_i(F_{j_0})} \times \Phi_{(F_j)_{j\in J'},i}\right) = \Phi_{(F_j)_{j\in J},i}$$

it follows from the induction hypothesis that $\Phi_{(F_j)_{j\in J},i}$ is a group isomorphism.

The last assertion follows from $\psi_j \circ \varphi_j = id_{F_j}$ for every $j \in J$ and

$$\sum_{j\in J} \varphi_j \circ \psi_j = id_{\prod\limits_{j\in J} F_j}\,.$$

■

PROPOSITION 1.3.4 *Let* $(F_j \overset{\phi_j}{\to} F'_j)_{j\in J}$ *be a finite family of morphisms in* $\mathfrak{M}_E$*,*

$$F := \prod_{j\in J} F_j\,, \qquad F' := \prod_{j\in J} F'_j\,,$$

and for every $j \in J$ *let*

$$\varphi_j : F_j \longrightarrow F\,, \qquad \varphi'_j : F'_j \longrightarrow F'$$

be the inclusion maps. Then the diagram

$$\begin{array}{ccc} \prod\limits_{j\in J} K_i(F_j) & \xrightarrow{\sum\limits_{j\in J} K_i(\varphi_j)} & K_i(F) \\ {\scriptstyle \prod\limits_{j\in J} K_i(\phi_j)}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i\left(\prod\limits_{j\in J}\phi_j\right)} \\ \prod\limits_{j\in J} K_i\left(F'_j\right) & \xrightarrow[\sum\limits_{j\in J} K_i\left(\varphi'_j\right)]{} & K_i(F') \end{array}$$

is commutative.

For every $j \in J$ the diagram

$$\begin{array}{ccc} F_j & \xrightarrow{\varphi_j} & F \\ {\scriptstyle \phi_j}\Big\downarrow & & \Big\downarrow{\scriptstyle \prod\limits_{j\in J}\phi_j} \\ F'_j & \xrightarrow[\varphi'_j]{} & F' \end{array}$$

is commutative so the diagram

$$\begin{array}{ccc} K_i(F_j) & \xrightarrow{K_i(\varphi_j)} & K_i(F) \\ {\scriptstyle K_i(\phi_j)}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i\left(\prod\limits_{j\in J}\phi_j\right)} \\ K_1(F'_j) & \xrightarrow[K_i\left(\varphi'_j\right)]{} & K_i(F') \end{array}$$

is also commutative. For $(a_j)_{j\in J} \in \prod\limits_{j\in J} K_i(F_j)$, by the above,

$$K_i\left(\prod_{j\in J}\phi_j\right) \circ \left(\sum_{j\in J} K_i(\varphi_j)\right)(a_j)_{j\in J} = K_i\left(\prod_{j\in J}\phi_j\right)\sum_{j\in J} K_i(\varphi_j)\,a_j =$$

$$= \sum_{j\in J} K_i \left(\prod_{k\in J} \phi_k \right) K_i(\varphi_j)\, a_j = \sum_{j\in J} K_i\left(\varphi'_j\right) K_i(\phi_j)\, a_j =$$

$$= \left(\sum_{j\in J} K_i\left(\varphi'_j\right) \right) (K_i(\phi_j)\, a_j)_{j\in J} = \left(\sum_{j\in J} K_i\left(\varphi'_j\right) \right) K_i \left(\prod_{j\in J} \phi_j \right) (a_j)_{j\in J} ,$$

which proves the assertion. ■

PROPOSITION 1.3.5

a) If $F \overset{\varphi}{\to} G,\ G \overset{\psi}{\to} F$ is a homotopy in $\mathfrak{M}_E$ then

$$K_i(\varphi) \circ K_i(\psi) = id_{K_i(G)}\,, \qquad K_i(\psi) \circ K_i(\varphi) = id_{K_i(F)}\,.$$

b) If F and G are homotopic E-C-algebras then $K_i(F)$ and $K_i(G)$ are isomorphic.*

c) If the E-C-algebra F is null-homotopic then it is K-null.*

a) follows from the homotopy axiom (Axiom 1.2.5).

b) follows from a).

c) follows from b) and from the null-axiom (Axiom 1.2.2). ■

PROPOSITION 1.3.6 *Let*

$$0 \longrightarrow F \overset{\varphi}{\longrightarrow} G \overset{\psi}{\longrightarrow} H \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$.

a) If F (resp. H) is K-null then

$$K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \qquad (\text{resp. } K_i(F) \xrightarrow{K_i(\varphi)} K_i(G))$$

is a group isomorphism.

b) If G is K-null then

$$K_i(H) \overset{\delta_i}{\longrightarrow} K_{i+1}(F)$$

is a group isomorphism.

c) If φ is K-null then the sequences

$$0 \longrightarrow K_i(G) \xrightarrow{K_i(\psi)} K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \longrightarrow 0$$

is exact.

d) If ψ is K-null then the sequences

$$0 \longrightarrow K_i(H) \xrightarrow{\delta_i} K_{i+1}(F) \xrightarrow{K_{i+1}(\varphi)} K_{i+1}(G) \longrightarrow 0$$

is exact.

e) The index maps of a split exact sequence are equal to 0.

a), b), c), and d) follow from the six-term axiom (Axiom 1.2.7).

e) follows from the six-term axiom (Axiom 1.2.7) and from the split exact axiom (Axiom 1.2.3). ■

PROPOSITION 1.3.7 *An $\mathfrak{M}_E$* **-triple** *is a triple (F_1, F_2, F_3) such that F_1 is an E-C*-algebra, F_2 is an E-ideal of F_1, and F_3 is an E-ideal of F_1 and of F_2. We denote for all $j,k \in \mathbb{N}_3$, $j<k$, by $\varphi_{j,k} : F_k \longrightarrow F_j$ the inclusion map, by $\psi_{j,k} : F_j \longrightarrow F_j/F_k$ the quotient map, and by $\delta_{j,k,i} : K_i(F_j/F_k) \longrightarrow F_k$ the index maps associated to the exact sequence in $\mathfrak{M}_E$*

$$0 \longrightarrow F_k \xrightarrow{\varphi_{j,k}} F_j \xrightarrow{\psi_{j,k}} F_j/F_k \longrightarrow 0 \,.$$

a) There is a unique morphism $F_2/F_3 \xrightarrow{\varphi_{1,2}/F_3} F_1/F_3$ in $\mathfrak{M}_E$ such that

$$\psi_{1,3} \circ \varphi_{1,2} = (\varphi_{1,2}/F_3) \circ \psi_{2,3} \,.$$

b) The diagram

$$\begin{array}{ccccccc}
K_i(F_3) & \xrightarrow{K_i(\varphi_{1,3})} & K_i(F_1) & \xrightarrow{K_i(\psi_{1,3})} & K_i(F_1/F_3) & \xrightarrow{\delta_{1,3,i}} & K_{i+1}(F_3) \\
= \uparrow & & K_i(\varphi_{1,2}) \uparrow & & \uparrow K_i(\varphi_{1,2}/F_3) & & \uparrow = \\
K_i(F_3) & \xrightarrow[K_i(\varphi_{2,3})]{} & K_i(F_2) & \xrightarrow[K_i(\psi_{2,3})]{} & K_i(F_2/F_3) & \xrightarrow[\delta_{2,3,i}]{} & K_{i+1}(F_3)
\end{array}$$

is commutative.

a) is easy to see.

b) follows from a), $\varphi_{1,2}\circ\varphi_{2,3}=\varphi_{1,3}$, and from the axiom of commutativity of the index maps (Axiom 1.2.8). ■

THEOREM 1.3.8 (The triple theorem) *Let (F_1,F_2,F_3) be an $\mathfrak{M}_E$-triple.*

a) Assume F_2 K-null.

a_1) $\delta_{2,3,i}:K_i(F_2/F_3)\longrightarrow K_{i+1}(F_3)$ is a group isomorphism.

a_2) $\delta_{2,3,i}=\delta_{1,3,i}\circ K_i(\varphi_{1,2}/F_3)$.

a_3) $\varphi_{1,3}$ is K-null.

a_4) If we put $\Phi_i:=K_i(\varphi_{1,2}/F_3)\circ(\delta_{2,3,i})^{-1}$ then

$$0\longrightarrow K_i(F_1)\xrightarrow{K_i(\psi_{1,3})}K_i(F_1/F_3)\underset{\xleftarrow{\Phi_i}}{\xrightarrow{\delta_{1,3,i}}}K_{i+1}(F_3)\longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1)\times K_{i+1}(F_3)\longrightarrow K_i(F_1/F_3),\quad (a,b)\longmapsto K_i(\psi_{1,3})a+\Phi_i b$$

is a group isomorphism.

b) Assume F_1/F_3 K-null.

b_1) $\delta_{2,3,i}=0$ and the sequence

$$0\longrightarrow K_i(F_3)\xrightarrow{K_i(\varphi_{2,3})}K_i(F_2)\xrightarrow{K_i(\psi_{2,3})}K_i(F_2/F_3)\longrightarrow 0$$

is exact.

b_2) $K_i(\varphi_{1,3}):K_i(F_3)\longrightarrow K_i(F_1)$ is a group isomorphism.

b_3) If we put $\Phi_i:=K_i(\varphi_{1,3})^{-1}\circ K_i(\varphi_{1,2})$ then the map

$$\Psi:K_i(F_2)\longrightarrow K_i(F_3)\times K_i(F_2/F_3),\quad b\longmapsto(\Phi_i b,K_i(\psi_{2,3})b)$$

is a group isomorphism.

b_4) If $\psi_{1,2}$ is K-null and if we put $\Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1}$ then

$$0 \longrightarrow K_{i+1}(F_1/F_2) \xrightarrow{\delta_{1,2,(i+1)}} K_i(F_2) \underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}} K_i(F_1) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(F_1) \times K_{i+1}(F_1/F_2) \longrightarrow K_i(F_2), \quad (a,b) \longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

is a group isomorphism.

c) Assume F_1 K-null and denote by ψ the canonical map $F_1/F_3 \to F_1/F_2$.

c_1) $\delta_{1,2,i}$ and $\delta_{1,3,i}$ are group isomorphisms.

c_2) $K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi)$.

c_3) Let $\varphi : F_1/F_2 \longrightarrow F_1/F_3$ be a morphism in $\mathfrak{M}_E$ such that

$$K_i(\psi \circ \varphi) = id_{K_i(F_1/F_2)} \,.$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3}) \circ \Phi_i = id_{K_i(F_2)}$. If in addition $\psi_{2,3}$ is K-null then

$$0 \longrightarrow K_{i+1}(F_2/F_3) \xrightarrow{\delta_{2,3,(i+1)}} K_i(F_3) \underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}} K_i(F_2) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(F_2/F_3) \times K_i(F_2) \longrightarrow K_i(F_3), \quad (a,b) \longmapsto \delta_{2,3,(i+1)} a + \Phi_i b$$

is a group isomorphism.

a_1) follows from Proposition 1.3.6 b).

a_2) follows from Proposition 1.3.7 b).

a_3) $\varphi_{1,3}$ factorizes through null and so it is K-null.

a_4) By a_2),

$$\delta_{1,3,i} \circ \Phi_i = \delta_{1,3,i} \circ K_i(\varphi_{1,2}/F_3) \circ (\delta_{2,3,i})^{-1} = \delta_{2,3,i} \circ (\delta_{2,3,i})^{-1} = id_{K_i(F_3)}$$

and this implies the assertion.

b_1) By Proposition 1.3.7 b), $\delta_{2,3,i}$ factorizes through null and so it is K-null. By the six-term axiom (Axiom 1.2.7) the sequence

$$0 \longrightarrow K_i(F_3) \xrightarrow{K_i(\varphi_{2,3})} K_i(F_2) \xrightarrow{K_i(\psi_{2,3})} K_i(F_2/F_3) \longrightarrow 0$$

is exact.

b_2) follows from Proposition 1.3.6 a).

b_3) Step 1

$$\Phi_i \circ K_i(\varphi_{2,3}) = id_{K_i(F_3)}$$

Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$\Phi_i \circ K_i(\varphi_{2,3}) = K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) =$$

$$= K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,3}) = id_{K_i(F_3)} .$$

Step 2 Ψ is injective

Let $b \in K_i(F_2)$ with $\Psi b = 0$. Then $K_i(\psi_{2,3})b = 0$ so by b_1),

$$b \in Ker\, K_i(\psi_{2,3}) = Im\, K_i(\varphi_{2,3})$$

and there is an $a \in K_i(F_3)$ with $b = K_i(\varphi_{2,3})a$. By Step 1,

$$a = \Phi_i K_i(\varphi_{2,3})a = \Phi_i b = 0\,,$$

so $b = 0$ and Ψ is injective.

Step 3 Ψ is surjective

Let $(a,c) \in K_i(F_3) \times K_i(F_2/F_3)$. Put $b' := K_i(\varphi_{2,3})a$. By b_1),

$$K_i(\psi_{2,3})b' = K_i(\psi_{2,3})K_i(\varphi_{2,3})a = 0$$

and by Step 1, $\Phi_i b' = \Phi_i K_i(\varphi_{2,3})a = a$. By b_1), there is a $b'' \in K_i(F_2)$ with $c = K_i(\psi_{2,3})b''$. By Step 1,

$$\Phi_i(b'' - K_i(\varphi_{2,3})\Phi_i b'') = \Phi_i b'' - \Phi_i K_i(\varphi_{2,3})\Phi_i b'' = \Phi_i b'' - \Phi_i b'' = 0 \,.$$

Thus by b_1),

$$\Psi(b' + b'' - K_i(\varphi_{2,3})\Phi_i b'') =$$

$$= (\Phi_i b', K_i(\psi_{2,3})b'' - K_i(\psi_{2,3})K_i(\varphi_{2,3})\Phi_i b'') = (a,c)$$

and Ψ is surjective.

b_4) Since $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3}$,

$$K_i(\varphi_{1,2}) \circ \Phi_i' = K_i(\varphi_{1,2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} =$$

$$= K_i(\varphi_{1,3}) \circ K_i(\varphi_{1,3})^{-1} = id_{K_i(F_1)}$$

and the assertion follows.

c_1) follows from Proposition 1.3.6 b)).

c_2) follows from the commutativity of the index maps (Axiom 1.2.8).

c_3) By c_2),

$$K_i(\varphi_{2,3}) \circ \Phi_i = K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} =$$

$$= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi) \circ K_{i+1}(\varphi) \circ (\delta_{1,2,(i+1)})^{-1} =$$

$$= \delta_{1,2,(i+1)} \circ K_{i+1}(\psi \circ \varphi) \circ (\delta_{1,2,(i+1)})^{-1} = \delta_{1,2,(i+1)} \circ (\delta_{1,2,(i+1)})^{-1} = id_{K_i(F_2)} \,.$$

The last assertion follows from the first one. ■

Remark. a) still holds with the weaker assumption that F_2 is only an E-C*-subalgebra of F_1.

1.4 Tensor Products

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.4.1 *Let G be a C*-algebra. We denote by $F \otimes G$ the spatial tensor product of F and G endowed with the structure of an E-C*-algebra by using the exterior multiplication*

$$E \times (F \otimes G) \longrightarrow F \otimes G, \quad (\alpha, x \otimes y) \longmapsto (\alpha x) \otimes y$$

([5] Proposition T.5.14 and T.5.17 Remark). *If $F \xrightarrow{\varphi} F'$ is a morphism in $\mathfrak{M}_E$ and $G \xrightarrow{\psi} G'$ a morphism in $\mathfrak{M}_{\mathbf{C}}$ then $F \otimes G \xrightarrow{\varphi \otimes \psi} F' \otimes G'$ denotes the morphism in $\mathfrak{M}_E$ defined by*

$$\varphi \otimes \psi : F \otimes G \longrightarrow F' \otimes G', \quad x \otimes y \longmapsto \varphi x \otimes \psi y\,.$$

If $(G_j)_{j \in J}$ is a family of C-algebras then we put*

$$\bigotimes_{j \in \emptyset} G_j := \mathbf{C}\,.$$

We have $F \otimes \mathbf{C} \approx F$ and $id_F \otimes id_G = id_{F \otimes G}$. If $F \xrightarrow{\varphi} F' \xrightarrow{\varphi'} F''$ are morphisms in $\mathfrak{M}_E$ and $G \xrightarrow{\psi} G' \xrightarrow{\psi'} G''$ are morphisms in $\mathfrak{M}_{\mathbf{C}}$ then

$$(\varphi \otimes \psi) \circ (\varphi' \otimes \psi') = (\varphi \circ \varphi') \otimes (\psi \circ \psi')\,.$$

If G and H are C*-algebras then

$$F \otimes (G \times H) \approx (F \otimes G) \times (F \otimes H), \qquad F \otimes (G \otimes H) \approx (F \otimes G) \otimes H\,.$$

If G is a C*-algebra and F_1, F_2 are E-C*-algebras then

$$(F_1 \times F_2) \otimes G \approx (F_1 \otimes G) \times (F_2 \otimes G)\,.$$

PROPOSITION 1.4.2 *Let G, H be C*-algebras.*

a) If $\varphi_0, \varphi_1 : G \longrightarrow H$ are homotopic C-homomorphisms then $id_F \otimes \varphi_0$ and $id_F \otimes \varphi_1$ are also homotopic.*

b) If $G \xrightarrow{\varphi} H$, $H \xrightarrow{\psi} G$ is a homotopy in $\mathfrak{M}_{\mathbf{C}}$ then

$$F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H\,, \qquad F \otimes H \xrightarrow{id_F \otimes \psi} F \otimes G$$

is a homotopy in $\mathfrak{M}_E$.

c) If G is homotopic to 0 then $F \otimes G$ is also homotopic to 0 and so K-null.

a) Let $[0,1] \longrightarrow \varphi_s$ be a pointwise continuous map of C*-homomorphisms $G \to H$. Let $z \in F \odot G$. There are finite families $(x_j)_{j\in J}$ in F and $(y_j)_{j\in J}$ in G such that

$$z = \sum_{j\in J} x_j \otimes y_j \ .$$

For $s \in [0,1]$,

$$(id_F \otimes \varphi_s)z = \sum_{j\in J} x_j \otimes \varphi_s y_j$$

so the map

$$[0,1] \longrightarrow F \otimes H, \quad s \longmapsto (id_F \otimes \varphi_s)z$$

is continuous.

Let now $z \in F \otimes G$, $s_0 \in [0,1]$, and $\varepsilon > 0$. There is a $z' \in F \odot G$ such that $\|z - z'\| < \frac{\varepsilon}{3}$. By the above, there is a $\delta > 0$ such that

$$\left\|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\right\| < \frac{\varepsilon}{3}$$

for all $s \in [0,1]$, $|s - s_0| < \delta$. It follows

$$\left\|(id_F \otimes \varphi_s)z - (id_F \otimes \varphi_{s_0})z\right\| \leq \left\|(id_F \otimes \varphi_s)(z - z')\right\| +$$

$$+ \left\|(id_F \otimes \varphi_s)z' - (id_F \otimes \varphi_{s_0})z'\right\| + \left\|(id_F \otimes \varphi_{s_0})(z - z')\right\| < \varepsilon \,,$$

which proves the assertion.

b) follows from a).

c) follows from b) and Proposition 1.3.5 c)). ■

PROPOSITION 1.4.3 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \underset{\lambda}{\overset{\psi}{\rightleftarrows}} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_{\mathbf{C}}$.

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \underset{id_F \otimes \lambda}{\overset{id_F \otimes \psi}{\rightleftarrows}} F \otimes G_3 \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \underset{K_i(id_F \otimes \lambda)}{\overset{K_i(id_F \otimes \psi)}{\rightleftarrows}} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact and the map

$$K_i(F \otimes G_1) \times K_i(F \otimes G_3) \longrightarrow K_i(F \otimes G_2),$$

$$(a,b) \longmapsto K_i(id_F \otimes \varphi)a + K_i(id_F \otimes \lambda)b$$

is a group isomorphism.

a) By [5] Corollary T.5.19, $id_F \otimes \varphi$ is injective. We have

$$(id_F \otimes \psi) \circ (id_F \otimes \lambda) = id_F \otimes (\psi \circ \lambda) = id_F \otimes id_{G_3} = id_{F \otimes G_3},$$

$$(id_F \otimes \psi) \circ (id_F \circ \varphi) = id_F \otimes (\psi \circ \varphi) = 0,$$

so

$$Im\,(id_F \otimes \varphi) \subset Ker\,(id_F \otimes \psi)\ .$$

Let $z \in (F \odot G_2) \cap Ker\,(id_F \otimes \psi)$. There is a linearly independent finite family $(x_j)_{j \in J}$ in F and a family $(y_j)_{j \in J}$ in G_2 such that

$$z = \sum_{j \in J} x_j \otimes y_j\ .$$

From

$$0 = (id_F \otimes \psi)z = \sum_{j \in J} x_j \otimes \psi y_j$$

we get $\psi y_j = 0$ for all $j \in J$. Thus for every $j \in J$ there is a $y'_j \in G_1$ with $\varphi y'_j = y_j$. It follows

$$z = \sum_{j \in J} x_j \otimes \varphi y'_j = (id_F \otimes \varphi) \sum_{j \in J} x_j \otimes y'_j \in Im\,(id_F \otimes \varphi)\ .$$

Let $z \in Ker\,(id_F \otimes \psi)$. Then

$$(id_F \otimes (\lambda \circ \psi))z = (id_F \otimes \lambda)(id_F \otimes \psi)z = 0\ .$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $F \odot G_2$ converging to z. For $n \in \mathbb{N}$, by the above,

$$(id_F \otimes \psi)(z_n - (id_F \otimes (\lambda \circ \psi))z_n) =$$

$$= (id_F \otimes \psi)z_n - (id_F \otimes \psi)(id_F \otimes \lambda)(id_F \otimes \psi)z_n =$$

$$= (id_F \otimes \psi)z_n - (id_F \otimes \psi)z_n = 0\,,$$

$$z_n - (id_F \otimes (\lambda \circ \psi))z_n \in Im\,(id_F \otimes \psi)\ .$$

Since $Im\,(id_F \otimes \varphi)$ is closed,

$$z = z - (id_F \otimes (\lambda \circ \psi))z = \lim_{n\to\infty} (z_n - (id_F \otimes (\lambda \circ \psi))z_n) \in Im\,(id_F \otimes \varphi)\,,$$

which proves the Proposition.

b) follows from a) and the split exact axiom (Axiom 1.2.3). ■

DEFINITION 1.4.4

We denote for every C-algebra G by* $\tilde{G}$ *its unitization* (see e.g. [4] Exercise 1.3) *and by*

$$0 \longrightarrow G \xrightarrow{\iota_G} \tilde{G} \underset{\lambda_G}{\overset{\pi_G}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

its associated split exact sequence. If G and H are C-algebras and* $\varphi : G \longrightarrow H$ *is a C*-homomorphism then* $\tilde{\varphi} : \tilde{G} \longrightarrow \tilde{H}$ *denotes the unitization of* φ.

COROLLARY 1.4.5 *Let G be a C*-algebra.*

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \iota_G} F \otimes \tilde{G} \underset{id_F \otimes \lambda_G}{\overset{id_F \otimes \pi_G}{\rightleftarrows}} F \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i\left(F \otimes \tilde{G}\right) \underset{K_i(id_F \otimes \lambda_G)}{\overset{K_i(id_F \otimes \pi_G)}{\rightleftarrows}} K_i(F) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(F \otimes G) \longrightarrow K_i\left(F \otimes \tilde{G}\right)\,,$$

$$(a,b) \longmapsto K_i\,(id_F \otimes \lambda_G)\,a + K_i\,(id_F \otimes \iota_G)\,b$$

is a group isomorphism.

c) Let $F \overset{\varphi}{\to} F'$ *be a morphism in* $\mathfrak{M}_E$ *and* $G \overset{\psi}{\to} G'$ *a morphism in* $\mathfrak{M}_{\mathbf{C}}$*. If we identify the isomorphic groups of b) then*

$$K_i(\varphi \otimes \tilde{\psi}) : K_i\left(F \otimes \tilde{G}\right) \longrightarrow K_i\left(F' \otimes \tilde{G}'\right),$$

$$(a,b) \longmapsto (K_i(\varphi)\,a, K_i(\varphi \otimes \psi)\,b)$$

is a group isomorphism.

d) Let $\varphi : G \longrightarrow G'$ *be a morphism in* $\mathfrak{M}_{\mathbf{C}}$*. If we denote by* Ψ_i *and* Ψ_i' *the group isomorphisms of b) associated to* G *and* G'*, respectively, then*

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi_i' \circ \left(id_{K_i(F)} \times K_i(id_F \otimes \varphi)\right) .$$

a) and b) follow from Proposition 1.4.3 a),b).

c) follows from b) and the commutativity of the following diagram:

$$\begin{array}{ccccc}
F \otimes G & \xrightarrow{id_F \otimes \iota_G} & F \otimes \tilde{G} & \xleftarrow{id_F \otimes \lambda_G} & F \otimes \mathbf{C} \\
{\scriptstyle \varphi \otimes \psi}\downarrow & & {\scriptstyle \varphi \otimes \tilde{\psi}}\downarrow & & \downarrow{\scriptstyle \varphi \otimes id_{\mathbf{C}}} \\
F' \otimes G' & \xrightarrow[id_{F'} \otimes \iota_{G'}]{} & F' \otimes \tilde{G}' & \xleftarrow[id_{F'} \otimes \lambda_{G'}]{} & F' \otimes \mathbf{C}
\end{array} .$$

d) For $(a,b) \in K_i(F) \times K_i(F \otimes G)$, since $\tilde{\varphi} \circ \lambda_G = \lambda_{G'}$ and $\iota_{G'} \circ \varphi = \tilde{\varphi} \circ \iota_G$,

$$K_i(id_F \otimes \tilde{\varphi})\,\Psi_i(a,b) = K_i(id_F \otimes \tilde{\varphi})\,(K_i(id_F \otimes \lambda_G)\,a + K_i(id_F \otimes \iota_G)\,b) =$$

$$= K_i(id_F \otimes \tilde{\varphi})\,K_i(id_F \otimes \lambda_G)\,a + K_i(id_F \otimes \tilde{\varphi})\,K_i(id_F \otimes \iota_G)\,b =$$

$$= K_i(id_F \otimes (\tilde{\varphi} \circ \lambda_G))\,a + K_i(id_F \otimes (\tilde{\varphi} \circ \iota_G))\,b =$$

$$= K_i(id_F \otimes \lambda_{G'})\,a + K_i(id_F \otimes (\iota_{G'} \circ \varphi))\,b =$$

$$= K_i(id_F \otimes \lambda_{G'})\,a + K_i(id_F \otimes \iota_{G'})\,K_i(id_F \otimes \varphi)\,b =$$

$$= \Psi_i'(a, K_i(id_F \otimes \varphi)\,b) = \Psi_i'(id_{K_i(F)} \times K_i(id_F \otimes \varphi))(a,b),$$

so

$$K_i(id_F \otimes \tilde{\varphi}) \circ \Psi_i = \Psi_i' \circ \left(id_{K_i(F)} \times K_i(id_F \otimes \varphi)\right) . \qquad \blacksquare$$

PROPOSITION 1.4.6 *If $(G_j)_{j\in J}$ is a finite family of C*-algebras then*

$$K_i\left(F\otimes\left(\bigotimes_{j\in J}\tilde{G}_j\right)\right)\approx\prod_{I\subset J}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right).$$

We prove the assertion by induction with respect to $Card\,J$. The assertion is trivial for $Card\,J=0$ (Definition 1.4.1 and Null-axiom (Axiom 1.4.6)). Let $j_0\in J$, $J':=J\setminus\{\mathrm{j}_0\}$, and assume the assertion holds for J'. By Corollary 1.4.5 b),

$$K_i\left(F\otimes\left(\bigotimes_{j\in J}\tilde{G}_j\right)\right)\approx K_i\left(\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\otimes\tilde{G}_{j_0}\right)\approx$$

$$\approx K_i\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\times K_i\left(\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\otimes G_{j_0}\right)\approx$$

$$\approx K_i\left(F\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\times K_i\left((F\otimes G_{j_0})\otimes\left(\bigotimes_{j\in J'}\tilde{G}_j\right)\right)\approx$$

$$\approx\prod_{I\subset J'}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right)\times\prod_{I\subset J'}K_i\left(F\otimes\left(\bigotimes_{j\in I\cup\{\mathrm{j}_0\}}G_j\right)\right)\approx$$

$$\approx\prod_{I\subset J}K_i\left(F\otimes\left(\bigotimes_{j\in I}G_j\right)\right).$$ ■

COROLLARY 1.4.7 *If G is a C*-algebra then for all $n\in\mathbb{N}^*$*

$$K_i\left(F\otimes\left(\bigotimes_{j\in\mathbb{N}_n}\tilde{G}\right)\right)\approx\prod_{k=0}^{n}K_i\left(F\otimes\left(\bigotimes_{j\in\mathbb{N}_k}G\right)\right)^{\binom{n}{k}}.$$ ■

PROPOSITION 1.4.8 *Let G be a C*-algebra and*

$$0\longrightarrow F_1\stackrel{\varphi}{\longrightarrow}F_2\underset{\lambda}{\overset{\psi}{\rightleftarrows}}F_3\longrightarrow 0$$

a split exact sequence in $\mathfrak{M}_E$.

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi \otimes id_G} F_2 \otimes G \underset{\lambda \otimes id_G}{\overset{\psi \otimes id_G}{\rightleftarrows}} F_3 \otimes G \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F_1 \otimes G) \xrightarrow{K_i(\varphi \otimes id_G)} K_i(F_2 \otimes G) \underset{K_i(\lambda \otimes id_G)}{\overset{K_i(\psi \otimes id_G)}{\rightleftarrows}} K_i(F_3 \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(F_1 \otimes G) \times K_i(F_3 \otimes G) \longrightarrow K_i(F_2 \otimes G),$$

$$(a,b) \longmapsto K_i(\varphi \otimes id_G)a + K_i(\lambda \otimes id_G)b$$

is a group isomorphism.

The proof is similar to the proof of Proposition 1.4.3. ■

PROPOSITION 1.4.9 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$*. If* F *or* G_3 *is nuclear then the sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact and so

$$\frac{F \otimes G_2}{F \otimes G_1} \approx F \otimes \frac{G_2}{G_1} .$$

[5] Theorem T.6.26. ■

PROPOSITION 1.4.10 *Let* G *be a C*-algebra and*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in $\mathfrak{M}_E$*. If* F_3 *or* G *is nuclear then*

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_2 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact.

[5] Theorem T.6.26. ■

DEFINITION 1.4.11 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ *and* G *a C*-algebra. If* δ_i *denotes the index maps associated to the above exact sequence in* $\mathfrak{M}_E$ *and if the sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\phi_1 \otimes id_G} F_1 \otimes G \xrightarrow{\phi_2 \otimes id_G} F_3 \otimes G \longrightarrow 0$$

is exact (e.g. F_3 or G is nuclear ([5] T.6.26)) *then we denote by* $\delta_{G,i}$ *the index maps associated to this last exact sequence in* $\mathfrak{M}_E$ *.*

In this case the six-term sequence

$$\begin{array}{ccccc} K_0(F_1 \otimes G) & \xrightarrow{K_0(\phi_1 \otimes id_G)} & K_0(F_2 \otimes G) & \xrightarrow{K_0(\phi_2 \otimes id_G)} & K_0(F_3 \otimes G) \\ \delta_{G,1} \Big\uparrow & & & & \Big\downarrow \delta_{G,0} \\ K_1(F_3 \otimes G) & \xleftarrow[K_1(\phi_2 \otimes id_G)]{} & K_1(F_2 \otimes G) & \xleftarrow[K_1(\phi_1 \otimes id_G)]{} & K_1(F_1 \otimes G) \end{array}$$

is exact (by the six-term axiom (Axiom 1.2.7)).

COROLLARY 1.4.12 *Let* G *be a unital C*-algebra,*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

an exact sequence in $\mathfrak{M}_E$*, and* δ_i *its index maps. We assume that* F_3 *or* G *is nuclear and put for every* $j \in \{1,2,3\}$

$$\varphi_j : F_j \longrightarrow F_j \otimes G, \quad x \longmapsto x \otimes 1_G\ .$$

Then $\delta_{G,i} \circ K_i(\varphi_3) = K_{i+1}(\varphi_1) \circ \delta_i$*.*

The diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 \\ \varphi_1 \Big\downarrow & & \varphi_2 \Big\downarrow & & \Big\downarrow \varphi_3 \\ F_1 \otimes G & \xrightarrow[\phi_1 \otimes id_G]{} & F_2 \otimes G & \xrightarrow[\phi_2 \otimes id_G]{} & F_3 \otimes G \end{array}$$

is commutative and the assertion follows from Proposition 1.4.10 and the commutativity of the index maps (Axiom 1.2.8). ■

1.5 The Class Υ

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.5.1 *Let Υ be the class of those C*-algebras G for which there are $p(G), q(G) \in \mathbb{N}^*$ and group isomorphisms*

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

such that for every morphism $F \xrightarrow{\phi} F'$ in $\mathfrak{M}_E$ the diagram

$$\begin{array}{ccc} K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i(F \otimes G) \\ {\scriptstyle K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)}} \Big\downarrow & & \Big\downarrow {\scriptstyle K_i(\phi \otimes id_G)} \\ K_i(F')^{p(G)} \times K_{i+1}(F')^{q(G)} & \xrightarrow[\Phi_{i,G,F'}]{} & K_i(F' \otimes G) \end{array}$$

is commutative. We denote by $\vec{G}$ the class of group isomorphisms

$$\Phi_{i,G,F} : K_i(F)^{p(G)} \times K_{i+1}(F)^{q(G)} \longrightarrow K_i(F \otimes G)$$

having the above property. A C-algebra G is called* **Υ-null** *if $G \in \Upsilon$ and $p(G) = q(G) = 0$.*

If G is Υ-null or if F is K-null and $G \in \Upsilon$ then $F \otimes G$ is K-null. In general we shall use $\Phi_{i,G,F}$ without writing $\{\Phi_{\mathrm{i,G,F}}\} \in \vec{G}$.

PROPOSITION 1.5.2 *Let $p.q \in \mathbb{N}^*$ and let Λ be the class of group isomorphisms*

$$\Lambda_{i,F} : K_i(F)^p \times K_{i+1}(F)^q \longrightarrow K_i(F)^p \times K_{i+1}(F)^q$$

such that for all morphisms $F \xrightarrow{\phi} F'$ in $\mathfrak{M}_E$ the diagram

$$\begin{array}{ccc} K_i(F)^p \times K_{i+1}(F)^q & \xrightarrow{\Lambda_{i,F}} & K_i(F)^p \times K_{i+1}(F)^q \\ {\scriptstyle K_i(\phi)^p \times K_{i+1}(\phi)^q} \Big\downarrow & & \Big\downarrow {\scriptstyle K_i(\phi)^p \times K_{i+1}(\phi)^q} \\ K_i(F')^p \times K_{i+1}(F')^q & \xrightarrow[\Lambda_{i,F'}]{} & K_i(\phi)^p \times K_{i+1}(\phi)^q \end{array}$$

is commutative. Let $G \in \Upsilon$ with $p(G) = p$, $q(G) = q$, and let $\{\Phi_{i,G,F}\} \in \vec{G}$.

a) If $\Lambda_{i,F} \in \Lambda$ *and if we put*

$$\Phi'_{i,G,F} := \Phi_{i,G,F} \circ \Lambda_{i,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(F \otimes G)$$

then $\{\Phi'_{i,G,F}\} \in \vec{G}$.

b) If $\{\Phi'_{i,G,F}\} \in \vec{G}$ *and if we put*

$$\Lambda_{i,F} := \Phi^{-1}_{i,G,F} \circ \Phi'_{i,G,F} : K_i(\phi)^p \times K_{i+1}(\phi)^q \longrightarrow K_i(\phi)^p \times K_{i+1}(\phi)^q$$

then $\{\Lambda_{i,F}\} \in \Lambda$.

c) If $\{\Lambda_{i,F}\}, \{\Lambda'_{\mathrm{i,F}}\} \in \Lambda$ *then* $\{\Lambda_{\mathrm{i,F}} \circ \Lambda'_{\mathrm{i,F}}\} \in \Lambda$, $\{\Lambda^{-1}_{\mathrm{i,F}}\} \in \Lambda$. ■

DEFINITION 1.5.3 *We denote for every nuclear* $G \in \Upsilon$ *by* G_Υ *the class of exact sequences in* $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

such that if δ_i *denote its index maps then the diagram*

$$\begin{array}{ccc} K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\Phi_{i,G,F_3}} & K_i(F_3 \otimes G) \\ \delta_i^{p(G)} \times \delta_{i+1}^{q(G)} \Big\downarrow & & \Big\downarrow \delta_{G,i} \\ K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} & \xrightarrow[\Phi_{(i+1),G,F_1}]{} & K_{i+1}(F_1 \otimes G) \end{array}$$

is commutative.

If G is Υ-null then every exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.

PROPOSITION 1.5.4

a) 0 *is* Υ*-null.*

b) $\mathbb{C} \in \Upsilon$, $p(\mathbb{C}) = 1$, $q(\mathbb{C}) = 0$, $\Phi_{i,\mathbb{C},F} = K_i(\phi_{\mathbb{C},F})$, *where*

$$\phi_{\mathbb{C},F} : F \longrightarrow F \times \mathbb{C}, \quad x \longmapsto x \otimes 1_{\mathbb{C}} .$$

Every exact sequence in $\mathfrak{M}_E$ *belongs to* $\mathbb{C}_\Upsilon$.

c) Let $G \xrightarrow{\varphi} G'$, $G' \xrightarrow{\psi} G$ *be a homotopy in* $\mathfrak{M}_{\mathbf{C}}$. *If* $G \in \Upsilon$ *then*

$$G' \in \Upsilon\,, \qquad p(G') = p(G)\,, \qquad q(G') = q(G)\,,$$

$$\Phi_{i,G',F} = K_i\left(id_F \otimes \varphi\right) \circ \Phi_{i,G,F}\ .$$

If in addition G *and* G' *are nuclear then* $G_\Upsilon = G'_\Upsilon$.

d) If G *is null-homotopic then* G *is* Υ*-null.*

a) By the null-axiom (Axiom 1.2.2), 0 is Υ-null.

b) The first assertion is easy to see. The second one follows from the commutativity of the index maps (Axiom 1.2.8).

c) By Proposition 1.4.2 b),

$$F \otimes G \overset{id_F \otimes \varphi}{\longrightarrow} F \otimes G'\,, \qquad F \otimes G' \overset{id_F \otimes \psi}{\longrightarrow} F \otimes G$$

is a homotopy in $\mathfrak{M}_E$. By Proposition 1.3.7 a),

$$K_i\left(id_F \otimes \varphi\right) : K_i\left(F \otimes G\right) \longrightarrow K_i\left(F \otimes G'\right)\,,$$

$$K_i\left(id_F \otimes \psi\right) : K_i\left(F \otimes G'\right) \longrightarrow K_i\left(F \otimes G\right)$$

are group isomorphisms and $K_i\left(id_F \otimes \psi\right) = K_i\left(id_F \otimes \varphi\right)^{-1}$. Thus

$$K_i\left(id_F \otimes \varphi\right) \circ \Phi_{i,G,F} : K_i\left(F\right)^{p(G)} \times K_{i+1}\left(F\right)^{q(G)} \longrightarrow K_i\left(F \otimes G'\right)$$

is a group isomorphism. If $F \xrightarrow{\phi} F'$ is a morphism in $\mathfrak{M}_E$ then the diagram

$$\begin{array}{ccccc}
K_i\left(F\right)^{p(G)} \times K_{i+1}\left(F\right)^{q(G)} & \xrightarrow{\Phi_{i,G,F}} & K_i\left(F \otimes G\right) & \xrightarrow{K_i\left(id_F \otimes \varphi\right)} & K_i\left(F \otimes G'\right) \\
\Big\downarrow K_i(\phi)^{p(G)} \times K_{i+1}(\phi)^{q(G)} & & \Big\downarrow K_i(\phi \otimes id_G) & & K_i\left(\phi \otimes id_{G'}\right) \Big\downarrow \\
K_i\left(F'\right)^{p(G)} \times K_{i+1}\left(F'\right)^{q(G)} & \xrightarrow[\Phi_{i,G,F'}]{} & K_i\left(F' \otimes G\right) & \xrightarrow[K_i\left(id_{F'} \otimes \varphi\right)]{} & K_i\left(F' \otimes G'\right)
\end{array}$$

is commutative and the first assertion follows.

Assume now that G and G' are nuclear, let

$$\left(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0\right) \in G_\Upsilon\,,$$

and let δ_i be its associated index maps. By the commutativity of the index maps (Axiom 1.2.8 a)) the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} & K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)} \\
\Phi_{i,G,F_3} \downarrow & & \downarrow \Phi_{(i+1),G,F_1} \\
K_i(F_3 \otimes G) & \xrightarrow[\delta_{G,i}]{} & K_{i+1}(F_1 \otimes G) \\
K_i(id_{F_3} \otimes \varphi) \downarrow & & \downarrow K_{i+1}(id_{F_1} \otimes \varphi) \\
K_i(F_3 \otimes G') & \xrightarrow[\delta_{G',i}]{} & K_{i+1}(F_1 \otimes G')
\end{array}$$

is commutative. Since the maps of the columns are group isomorphisms, it follows by the above, that the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G')} \times K_{i+1}(F_3)^{q(G')} & \xrightarrow{\delta_i^{p(G')} \times \delta_{i+1}^{q(G')}} & K_{i+1}(F_1)^{p(G')} \times K_i(F_1)^{q(G')} \\
\Phi_{i,G',F_3} \downarrow & & \downarrow \Phi_{(i+1),G',F_1} \\
K_i(F_3 \otimes G') & \xrightarrow[\delta_{G',i}]{} & K_{i+1}(F_1 \otimes G')
\end{array}$$

is also commutative.

d) follows from a) and c). ■

PROPOSITION 1.5.5 *Let G be a nuclear C*-algebra belonging to Υ.*

a) Every split exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.

b) Every exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

for which F_1 or F_3 is homotopic to 0 belongs to G_Υ .

a) Let

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_2 \underset{\underset{\lambda}{\longleftarrow}}{\overset{\psi}{\longrightarrow}} F_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_E$ and let δ_i be its index maps. By Proposition 1.4.8 a),

$$0 \longrightarrow F_1 \otimes G \xrightarrow{\varphi\otimes id_G} F_2 \otimes G \underset{\lambda\otimes id_G}{\overset{\psi\otimes id_G}{\rightleftarrows}} F_3 \otimes G \longrightarrow 0$$

is split exact and so by Proposition 1.3.1, $\delta_i = \delta_{G,i} = 0$.

b) By Proposition 1.4.2 c), $F_1 \otimes G$ or $F_3 \otimes G$ is null-homotopic and so K-null. Thus by the six-term axiom (Axiom 1.2.7), $\delta_i = \delta_{G,i} = 0$, where δ_i denote the index maps associated to

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0 \,. \qquad \blacksquare$$

PROPOSITION 1.5.6 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ such that G_3 is nuclear.

a) Assume G_1 is Υ-null.

a$_1$) $K_i(id_F \otimes \psi) : K_i(F \otimes G_2) \longrightarrow K_i(F \otimes G_3)$ is a group isomorphism.

a$_2$) If $G_2 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_2, G_3 \in \Upsilon\,, \qquad p(G_2) = p(G_3)\,, \qquad q(G_2) = q(G_3)\,,$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F} \,.$$

If in addition G_2 is nuclear then $(G_2)_\Upsilon = (G_3)_\Upsilon$.

b) Assume G_2 is Υ-null and let δ_i^F denote the index maps of the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F\otimes\varphi} F \otimes G_2 \xrightarrow{id_F\otimes\psi} F \otimes G_3 \longrightarrow 0 \,.$$

b$_1$) $\delta_i^F : K_i(F \otimes G_3) \longrightarrow K_{i+1}(F \otimes G_1)$ is a group isomorphism.

b$_2$) If $G_1 \in \Upsilon$ or $G_3 \in \Upsilon$ then

$$G_1, G_3 \in \Upsilon\,, \qquad p(G_1) = q(G_3)\,, \qquad q(G_1) = p(G_3)\,,$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F \,.$$

c) Assume G_3 is Υ-null.

c_1) $K_i(id_F \otimes \varphi) : K_i(F \otimes G_1) \longrightarrow K_i(F \otimes G_2)$ is a group isomorphism.

c_2) If $G_1 \in \Upsilon$ or $G_2 \in \Upsilon$ then

$$G_1, G_2 \in \Upsilon, \qquad p(G_1) = p(G_2), \qquad q(G_1) = q(G_2),$$

$$\Phi_{i,G_2,F} = K_i(id_F \otimes \varphi) \circ \Phi_{i,G_1,F} .$$

If in addition G_1 and G_2 are nuclear then $(G_1)_\Upsilon = (G_2)_\Upsilon$.

By Proposition 1.4.9 a), the sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \xrightarrow{id_F \otimes \psi} F \otimes G_3 \longrightarrow 0$$

is exact. If G_j is Υ-null then $F \otimes G_j$ is K-null so $a_1), b_1), c_1)$ follow from Proposition 1.3.6 a),b).

a_2) By a_1), it is easy to see that

$$G_2, G_3 \in \Upsilon, \qquad p(G_2) = p(G_3), \qquad q(G_2) = q(G_3),$$

$$\Phi_{i,G_3,F} = K_i(id_F \otimes \psi) \circ \Phi_{i,G_2,F} .$$

Assume now G_2 nuclear. Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

belong to $(G_2)_\Upsilon$ or $(G_3)_\Upsilon$ and let δ_i be its associated index maps. Consider the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \\
\Phi_{i,G_2,F_3} \downarrow & & \downarrow \Phi_{(i+1),G_2,F_1} \\
K_i(F_3 \otimes G_2) & \xrightarrow{\delta_{G_2,i}} & K_{i+1}(F_1 \otimes G_2) \\
K_i(id_{F_3} \otimes \psi) \downarrow & & \downarrow K_{i+1}(id_{F_1} \otimes \psi) \\
K_i(F_3 \otimes G_3) & \xrightarrow[\delta_{G_3,i}]{} & K_{i+1}(F_1 \otimes G_3) \\
\Phi_{i,G_3,F_3} \uparrow & & \uparrow \Phi_{(i+1),G_3,F_1} \\
K_i(F_3)^{p(G_3)} \times K_{i+1}(F_3)^{q(G_3)} & \xrightarrow[\delta_i^{p(G_3)} \times \delta_{i+1}^{q(G_3)}]{} & K_{i+1}(F_1)^{p(G_3)} \times K_i(F_1)^{q(G_3)} .
\end{array}$$

Its upper part or lower part is commutative and the maps of the columns are group isomorphisms. It follows, by the above, that the diagram is commutative. Thus $(G_2)_\Upsilon = (G_3)_\Upsilon$.

b_2) Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_i(F \otimes G_1) & \xrightarrow{K_i(id_F \otimes \varphi)} & K_i(F \otimes G_2) & \xrightarrow{K_i(id_F \otimes \psi)} & \\
 & & {\scriptstyle K_i(\phi \otimes id_{G_1})}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i(\phi \otimes id_{G_2})} & & \\
0 & \longrightarrow & K_i(F' \otimes G_1) & \xrightarrow[K_i(id_{F'} \otimes \varphi)]{} & K_i(F' \otimes G_2) & \xrightarrow[K_i(id_{F'} \otimes \psi)]{} &
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{K_i(id_F \otimes \varphi)} & K_i(F \otimes G_2) & \xrightarrow{K_i(id_F \otimes \psi)} & K_i(F \otimes G_3) & \longrightarrow & 0 \\
 & {\scriptstyle K_i(\phi \otimes id_{G_2})}\Big\downarrow & & \Big\downarrow{\scriptstyle K_i(\phi \otimes id_{G_3})} & & \\
\xrightarrow[K_i(id_{F'} \otimes \varphi)]{} & K_i(F' \otimes G_2) & \xrightarrow[K_i(id_{F'} \otimes \psi)]{} & K_i(F' \otimes G_3) & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{array}{ccc}
K_i(F \otimes G_3) & \xrightarrow{\delta_i^F} & K_{i+1}(F \otimes G_1) \\
{\scriptstyle K_i(\phi \otimes id_{G_3})}\Big\downarrow & & \Big\downarrow{\scriptstyle K_{i+1}(\phi \otimes id_{G_1})} \\
K_i(F' \otimes G_3) & \xrightarrow[\delta_i^{F'}]{} & K_{i+1}(F' \otimes G_1)
\end{array}$$

is commutative. By b_1),

$$G_1, G_3 \in \Upsilon, \qquad p(G_1) = q(G_3), \qquad q(G_1) = p(G_3),$$

$$\Phi_{i,G_3,F} = \Phi_{(i+1),G_1,F} \circ \delta_i^F \, .$$

c_2) The proof is similar to the proof of a_2). ■

PROPOSITION 1.5.7 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \underset{\lambda}{\overset{\psi}{\rightleftarrows}} G_3 \longrightarrow 0$$

be a split exact sequence in $\mathfrak{M}_{\mathbf{C}}$.

a) If $G_1, G_3 \in \Upsilon$ then

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + p(G_3), \qquad q(G_2) = q(G_1) + q(G_3),$$

$$\Phi_{i,G_2,F} = (K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)) \circ (\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}) .$$

b) If in addition G_1,G_2,and G_3 are nuclear then $(G_1)_\Upsilon \cap (G_3)_\Upsilon \subset (G_2)_\Upsilon$.

a) By Proposition 1.4.3 b), the sequence

$$0 \longrightarrow K_i(F \otimes G_1) \xrightarrow{K_i(id_F \otimes \varphi)} K_i(F \otimes G_2) \underset{K_i(id_F \otimes \lambda)}{\overset{K_i(id_F \otimes \psi)}{\rightleftarrows}} K_i(F \otimes G_3) \longrightarrow 0$$

is split exact. Thus the maps

$$\left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)}\right) \times \left(K_i(F)^{p(G_3)} \times K_{i+1}(F)^{q(G_3)}\right)$$

$$\xrightarrow{\Phi_{i,G_1,F} \times \Phi_{i,G_3,F}} K_i(F \otimes G_1) \times K_i(F \otimes G_3) \xrightarrow{K_i(id_F \otimes \varphi) \times K_i(id_F \otimes \lambda)} K_i(F \otimes G_2)$$

are group isomorphisms.

Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Since the diagram with split exact rows

$$0 \longrightarrow F \otimes G_1 \xrightarrow{id_F \otimes \varphi} F \otimes G_2 \underset{id_F \otimes \lambda}{\overset{id_F \otimes \psi}{\rightleftarrows}} F \otimes G_3 \longrightarrow 0,$$

$$0 \longrightarrow F' \otimes G_1 \xrightarrow{id_{F'} \otimes \varphi} F' \otimes G_2 \underset{id_{F'} \otimes \lambda}{\overset{id_{F'} \otimes \psi}{\rightleftarrows}} F' \otimes G_3 \longrightarrow 0,$$

(Proposition 1.4.3 a)) and with columns $\phi \otimes id_{G_1}$, $\phi \otimes id_{G_2}$, and $\phi \otimes id_{G_3}$ is commutative, the assertion follows from Proposition 1.4.3 b).

b) Let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_\Upsilon \cap (G_3)_\Upsilon$$

and let δ_i be its associated index maps. Consider the diagram (by a))

$$\begin{array}{ccc}
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)} \\
\Phi_{i,G_1,F_3} \times \Phi_{i,G_3,F_3} \downarrow & & \Phi_{(i+1),G_1,F_1} \times \Phi_{(i+1),G_3,F_1} \downarrow \\
K_i(F_3 \otimes G_1) \times K_i(F_3 \otimes G_3) & \xrightarrow[\delta_{G_1,i} \times \delta_{G_3,i}]{} & K_{i+1}(F_1 \otimes G_1) \times K_{i+1}(F_1 \otimes G_3) \\
A \downarrow & & K_{i+1}\left(id_{F_1} \otimes \varphi\right) \times K_{i+1}\left(id_{F_1} \otimes \lambda\right) \downarrow \\
K_i(F_3 \otimes G_2) & \xrightarrow[\delta_{G_2,i}]{} & K_{i+1}(F_1 \otimes G_2) \\
\Phi_{i,G_2,F_3} \uparrow & & \uparrow \Phi_{(i+1),G_2,F_1} \\
K_i(F_3)^{p(G_2)} \times K_{i+1}(F_3)^{q(G_2)} & \xrightarrow{\delta_i^{p(G_2)} \times \delta_{i+1}^{q(G_2)}} & K_{i+1}(F_1)^{p(G_2)} \times K_i(F_1)^{q(G_2)}\,,
\end{array}$$

where

$$A := K_i\left(id_{F_3} \otimes \varphi\right) \times K_i\left(id_{F_3} \otimes \lambda\right) \,.$$

Its upper part is commutative and the maps of the columns are group isomorphisms. It follows that the lower part of the diagram is also commutative. ■

COROLLARY 1.5.8 *If $G \in \Upsilon$ then $\tilde{G} \in \Upsilon$, $p(\tilde{G}) = p(G)+1$, $q(\tilde{G}) = q(G)$. If in addition G and $\tilde{G}$ are nuclear then $G_\Upsilon \subset \tilde{G}_\Upsilon$.* ■

PROPOSITION 1.5.9 *Let $(G_j)_{j\in J}$ be a finite family in Υ.*

a)

$$G := \prod_{j\in J} G_j \in \Upsilon\,, \qquad p(G) = \sum_{j\in J} p(G_j)\,, \qquad q(G) = \sum_{j\in J} q(G_j)\,,$$

$$\Phi_{i,G,F} = \left(\prod_{j\in J} \Phi_{i,G_j,F}\right) \circ \Phi_{(F\otimes G_j)_{j\in J},i}\,.$$

In particular if G_j is Υ-null for every $j \in J$ then G is Υ-null.

b) If in addition G and all G_j, $j \in J$, are nuclear then

$$\bigcap_{j\in J} (G_j)_\Upsilon \subset G_\Upsilon\,.$$

c) $\mathbf{C}^J \in \Upsilon$, $p(\mathbf{C}^J) = Card\, J$, $q(\mathbf{C}^J) = 0$, *and every exact sequence in* $\mathfrak{M}_E$ *belongs to* $\left(\mathbf{C}^J\right)_\Upsilon$.

a) We put

$$\bar{p} := \sum_{j \otimes J} p(G_j)\,, \qquad\qquad \bar{q} := \sum_{j \otimes J} q(G_j)\,.$$

Since

$$F \otimes \prod_{j \in J} G_j \approx \prod_{j \in J} (F \otimes G_j)\,,$$

by Proposition 1.3.3, the maps

$$\prod_{j \in J} \Phi_{i,G_j,F} : K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} = \prod_{j \in J} \left(K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} \right) \longrightarrow$$

$$\longrightarrow \prod_{j \in J} K_i(F \otimes G_j) \xrightarrow{\Phi_{(F \otimes G_j)_{j \in J}, i}} K_i(F \otimes G)$$

are group isomorphisms. Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. The diagram

$$\begin{array}{ccc}
K_i(F)^{\bar{p}} \times K_{i+1}(F)^{\bar{q}} & \xrightarrow{\prod_{j \in J} \Phi_{i,G_j,F}} & \prod_{j \in J} K_i(F \otimes G_j) \\
{\scriptstyle K_i(\phi)^{\bar{p}} \times K_{i+1}(\phi)^{\bar{q}}} \big\downarrow & & \big\downarrow {\scriptstyle \prod_{j \in J} K_i\left(\phi \otimes id_{G_j}\right)} \\
K_i(F')^{\bar{p}} \times K_{i+1}(F')^{\bar{q}} & \xrightarrow[\prod_{j \in J} \Phi_{i,G_j,F'}]{} & \prod_{j \in J} K_i(F' \otimes G_j)
\end{array}$$

is obviously commutative and by Proposition 1.3.4 the diagram

$$\begin{array}{ccc}
\prod_{j \in J} K_i(F \otimes G_j) & \xrightarrow{\Phi_{(F \otimes G_j)_{j \in J}, i}} & K_i(F \otimes G) \\
{\scriptstyle \prod_{j \in J} K_i\left(\phi \otimes id_{G_j}\right)} \big\downarrow & & \big\downarrow {\scriptstyle K_i(\phi \otimes id_G)} \\
\prod_{j \in J} K_i(F' \otimes G_j) & \xrightarrow[\Phi_{(F' \otimes G_j)_{j \in J}, i}]{} & K_i(F' \otimes G)
\end{array}$$

is also commutative and this proves the assertion.

b) follows from Proposition 1.5.7 by complete induction.

c) follows from a), b), and Proposition 1.5.4 b). ■

PROPOSITION 1.5.10 *Let J be a finite set and for every $j \in J$ let*

$$0 \longrightarrow F_{j,1} \xrightarrow{\varphi_j} F_{j,2} \xrightarrow{\psi_j} F_{j,3} \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$ and $\delta_{j,i}$ its associated index maps. For every $k \in \{1,2,3\}$ put

$$F_k := \prod_{j \in J} F_{j,k}$$

and for every $j \in J$ denote by

$$\varphi_{j,k} : F_{j,k} \longrightarrow F_k \,, \qquad \psi_{j,k} : F_k \longrightarrow F_{j,k}$$

the canonical inclusion and projection, respectively. Then

$$0 \longrightarrow F_1 \xrightarrow{\prod\limits_{j\in J} \varphi_j} F_2 \xrightarrow{\prod\limits_{j\in J} \psi_j} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ and if we denote by δ_i its index maps then the diagram

$$\begin{array}{ccc}
\prod\limits_{j\in J} K_i\left(F_{j,3}\right) & \xrightarrow{\Psi_{3,i}} & K_i\left(F_3\right) \\
{\scriptstyle \prod\limits_{j\in J} \delta_{j,i}} \big\downarrow & & \big\downarrow {\scriptstyle \delta_i} \\
\prod\limits_{j\in J} K_{i+1}\left(F_{j,1}\right) & \xrightarrow[\Psi_{1,(i+1)}]{} & K_{i+1}\left(F_1\right)
\end{array}$$

is commutative, where for every $k \in \{1,3\}$,

$$\Psi_{k,i} : \prod_{j\in J} K_i\left(F_{j,k}\right) \longrightarrow K_i\left(F_k\right) \,, \quad \left(a_j\right)_{j\in J} \longmapsto \sum_{j\in J} K_i\left(\varphi_{j,k}\right) a_j \,.$$

For every $j \in J$ the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{j,1} & \xrightarrow{\varphi_j} & F_{j,2} & \xrightarrow{\psi_j} & F_{j,3} & \longrightarrow & 0 \\
 & & {\scriptstyle \varphi_{j,1}} \big\downarrow & & {\scriptstyle \varphi_{j,2}} \big\downarrow & & \big\downarrow {\scriptstyle \varphi_{j,3}} & & \\
0 & \longrightarrow & F_1 & \xrightarrow[\prod\limits_{j\in J} \varphi_j]{} & F_2 & \xrightarrow[\prod\limits_{j\in J} \psi_j]{} & F_3 & \longrightarrow & 0\,,
\end{array}$$

is commutative. By the commutativity of the index maps (Axiom 1.2.8), the diagram

$$\begin{array}{ccc}
K_i\left(F_{j,3}\right) & \xrightarrow{K_i\left(\varphi_{j,3}\right)} & K_i\left(F_3\right) \\
{\scriptstyle \delta_{j,i}} \big\downarrow & & \big\downarrow {\scriptstyle \delta_i} \\
K_{i+1}\left(F_{j,1}\right) & \xrightarrow[K_{i+1}\left(\varphi_{j,1}\right)]{} & K_{i+1}\left(F_1\right)
\end{array}$$

is commutative. Let $(a_j)_{j\in J} \in \prod_{j\in J} K_i\left(F_{j,3}\right)$. Then

$$\delta_i \Psi_{3,i}(a_j)_{j\in J} = \delta_i \sum_{j\in J} K_i\left(\varphi_{j,3}\right) a_j =$$

$$= \sum_{j\in J} K_{i+1}\left(\varphi_{j,1}\right) \delta_{j,i} a_j = \Psi_{1,(i+1)}\left(\prod_{j\in J} \delta_{j,i}\right)(a_j)_{j\in J} .$$

Thus the diagram

$$\begin{array}{ccc} \prod_{j\in J} K_i\left(F_{j,3}\right) & \xrightarrow{\Psi_{3,i}} & K_i\left(F_3\right) \\ \prod_{j\in J}\delta_{j,i} \Big\downarrow & & \Big\downarrow \delta_i \\ \prod_{j\in J} K_{i+1}\left(F_{j,1}\right) & \xrightarrow[\Psi_{1,(i+1)}]{} & K_{i+1}\left(F_1\right) \end{array}$$

is commutative. ■

PROPOSITION 1.5.11 *Let $(G_j)_{j\in J}$ be a finite family in Υ.*

a)

$$G := \bigotimes_{j\in J} G_j \in \Upsilon ,$$

$$p(G) = \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j)) + \prod_{j\in J}(p(G_j)-q(G_j))\right) ,$$

$$q(G) = \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j)) - \prod_{j\in J}(p(G_j)-q(G_j))\right) .$$

b) If G_{j_0} is K-null for a $j_0 \in J$ then $F \otimes \left(\bigotimes_{j\in J} G_j\right)$ is also K-null.

c) If $p(G_{j_0}) = q(Gj_0)$ for a $j_0 \in J$ then $p(G) = q(G)$.

d) Let $j_0 \in J$, $J' := J \setminus \{j_0\}$, and $G' := \bigotimes_{j\in J'} G_j$.

d_1) If $p(G_{j_0}) = 1$, $q(G_{j_0}) = 0$ then $p(G') = p(G)$, $q(G') = q(G)$.

d_2) If $p(G_{j_0}) = 0$, $q(G_{j_0}) = 1$ then $p(G') = q(G)$, $q(G') = p(G)$.

e) If we put

$$H := \bigotimes_{j\in J} \tilde{G}_j \qquad \text{and} \qquad G_I := \bigotimes_{j\in I} G_j$$

for every $I \subset J$ then

$$H \in \Upsilon, \qquad p(H) = \sum_{I\subset J} p(G_I), \qquad q(H) = \sum_{I\subset J} q(G_I);.$$

f) If in addition G and all $(G_j)_{j\in J}$ are nuclear then

$$\bigcap_{j\in J} (G_j)_\Upsilon \subset G_\Upsilon \,.$$

a) Assume first $J = \{1,2\}$. The maps

$$K_i(F)^{p(G_1)p(G_2)+q(G_1)q(G_2)} \times K_{i+1}(F)^{p(G_1)q(G_2)+p(G_2)q(G_1)} =$$

$$= \left(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)}\right)^{p(G_2)} \times \left(K_{i+1}(F)^{p(G_1)} \times K_i(F)^{q(G_1)}\right)^{q(G_2)} \longrightarrow$$

$$\xrightarrow{(\Phi_{i,G_1,F})^{p(G_2)} \times (\Phi_{(i+1),G_1,F})^{q(G_2)}}$$

$$\longrightarrow K_i(F \otimes G_1)^{p(G_2)} \times K_{i+1}(F \otimes G_1)^{q(G_2)} \longrightarrow$$

$$\xrightarrow{\Phi_{i,G_2,F\otimes G_1}}$$

$$\longrightarrow K_i((F \otimes G_1) \otimes G_2) \approx K_i(F \otimes (G_1 \otimes G_2))$$

are group isomorphisms and

$$p(G_1 \otimes G_2) := p(G_1)p(G_2) + q(G_1)q(G_2) =$$

$$= \frac{1}{2}[(p(G_1)+q(G_1))(p(G_2)+q(G_2)) + (p(G_1)-q(G_1))(p(G_2)-q(G_2))],$$

$$q(G_1 \otimes G_2) := p(G_1)q(G_2) + p(G_2)q(G_1) =$$

$$= \frac{1}{2}[(p(G_1)+q(G_1))(p(G_2)+q(G_2)) - (p(G_1)-q(G_1))(p(G_2)-q(G_2))].$$

If $F \xrightarrow{\phi} F'$ is a morphism in $\mathfrak{M}_E$ then the diagrams

$$\begin{array}{ccc}
K_i(F\otimes G_1)^{p(G_2)} \times K_{i+1}(F\otimes G_1)^{q(G_2)} & \xrightarrow{\Phi_{i,G_2,(F\otimes G_1)}} & K_i((F\otimes G_1)\otimes G_2) \\
\Big\downarrow K_i\left(\phi\otimes id_{G_1}\right)^{p(G_2)} \times K_{i+1}\left(\varphi\otimes id_{G_1}\right)^{q(G_2)} & & \Big\downarrow K_i\left((\phi\otimes id_{G_1})\otimes id_{G_2}\right) \\
K_i(F'\otimes G_1)^{p(G_2)} \times K_{i+1}(F'\otimes G_1)^{q(G_2)} & \xrightarrow[\Phi_{i,G_2,(F'\otimes G_1)}]{} & K_i((F'\otimes G_1)\otimes G_2)
\end{array}$$

$$\begin{array}{ccc} K_i((F\otimes G_1)\otimes G_2) & \xrightarrow{\approx} & K_i(F\otimes(G_1\otimes G_2)) \\ \downarrow{\scriptstyle K_i((\phi\otimes id_{G_1})\otimes id_{G_2})} & & \downarrow{\scriptstyle K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right)} \\ K_i((F'\otimes G_1)\otimes G_2) & \xrightarrow[\approx]{} & K_i(F'\otimes(G_1\otimes G_2)) \end{array}$$

are commutative, which proves the assertion in this case.

The general case is obtained now by induction with respect to $Card\, J$. Let $Card\, J > 1$, $k \in J$, $J' := J \setminus \{\mathrm{k}\}$, $G' := \otimes_{j\in J'} G_j$ and assume the assertion holds for J'. By the above,

$$p(G) = \frac{1}{2}[(p(G')+q(G'))(p(G_k)+q(G_k))+(p(G')-q(G'))(p(G_k)-q(G_k))] =$$

$$= \frac{1}{2}\left(\prod_{j\in J'}(p(G_j)+q(G_j))(p(G_k)+q(G_k))+\right.$$

$$\left.+\prod_{j\in J'}(p(G_j)-q(G_j))(p(G_k)-q(G_k))\right) =$$

$$= \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j))+\prod_{j\in J}(p(G_j)-q(G_j))\right),$$

$$q(G) = \frac{1}{2}[(p(G')+q(G'))(p(G_k)+q(G_k))-(p(G')-q(G'))(p(G_k)-q(G_k))] =$$

$$= \frac{1}{2}\left(\prod_{j\in J'}(p(G_j)+q(G_j))(p(G_k)+q(G_k))-\right.$$

$$\left.-\prod_{j\in J'}(p(G_j)-q(G_j))(p(G_k)-q(G_k))\right) =$$

$$= \frac{1}{2}\left(\prod_{j\in J}(p(G_j)+q(G_j))-\prod_{j\in J}(p(G_j)-q(G_j))\right).$$

b), c), and d) follow directly from a).

e) By Corollary 1.5.8, $\tilde{G}_j \in \Upsilon$ for every $j \in J$. By a) and Proposition 1.4.6, $H \in \Upsilon$,

$$K_i(F\otimes H) \approx \prod_{I\subset J} K_i(F\otimes G_I) \approx \prod_{I\subset J}\left(K_i(F)^{p(G_I)} \times K_{i+1}(F)^{q(G_I)}\right) =$$

$$= K_i(F)^{\sum\limits_{I\subset J} p(G_I)} \times K_{i+1}(F)^{\sum\limits_{I\subset J} q(G_I)}.$$

f) Assume first $J := \{1,2\}$, let

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in (G_1)_\Upsilon \cap (G_2)_\Upsilon\,,$$

and let δ_i be its index maps. Then (by a)) the diagram

$$\begin{array}{ccc}
K_i(F_3)^{p(G)} \times K_{i+1}(F_3)^{q(G)} & \xrightarrow{\delta_i^{p(G)} \times \delta_{i+1}^{q(G)}} & A \\
{\scriptstyle \Phi^{p(G_2)}_{i,G_1,F_3} \times \Phi^{q(G_2)}_{(i+1),G_1,F_3}} \Big\downarrow & & \Big\downarrow {\scriptstyle \Phi^{p(G_2)}_{(i+1),G_1,F_1} \times \Phi^{q(G_2)}_{i,G_1,F_1}} \\
K_i(F_3 \otimes G_1)^{p(G_2)} \times K_{i+1}(F_3 \otimes G_1)^{q(G_2)} & \xrightarrow{\delta^{p(G_1)}_{G_1,i} \times \delta^{q(G_2)}_{G_1,i}} & B \\
{\scriptstyle \Phi_{i,G_2,(F_3 \otimes G_1)}} \Big\downarrow & & \Big\downarrow {\scriptstyle \Phi_{(i+1),G_2,(F_1 \otimes G_1)}} \\
K_i((F_3 \otimes G_1) \otimes G_2) \approx K_i(F_3 \otimes (G_1 \otimes G_2)) & \xrightarrow[\delta_{G_1,(G_2,i)}]{} & C
\end{array}$$

is commutative, where

$$A := K_{i+1}(F_1)^{p(G)} \times K_i(F_1)^{q(G)}\,,$$
$$B := K_{i+1}(F_1 \otimes G_1)^{p(G_2)} \times K_i(F_1 \otimes G_1)^{q(G_2)}\,,$$
$$C := K_{i+1}(((F_1 \otimes G_1) \otimes G_2)) \approx K_{i+1}((F_1 \otimes (G_1 \otimes G_2)))\,.$$

Thus

$$(0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0) \in G_\Upsilon\,.$$

The general case follows by induction with respect to *Card J*. ■

COROLLARY 1.5.12 *Let $G \in \Upsilon$, $n \in \mathbb{N}$, and $H := \otimes_{j \in \mathbb{N}_n} G$. Then $H \in \Upsilon$, $G_\Upsilon \subset H_\Upsilon$, and*

$$p(H) = \frac{1}{2}\left((p(G)+q(G))^n + (p(G)-q(G))^n\right)\,,$$
$$q(H) = \frac{1}{2}\left((p(G)+q(G))^n - (p(G)-q(G))^n\right)\,.$$

The assertion follows from Proposition 1.5.11 a). ■

PROPOSITION 1.5.13 *Let (G_1,G_2,G_3) be an $\mathfrak{M}_{\mathbf{C}}$-triple such that G_1/G_3 and G_2/G_3 are nuclear, G_2 is Υ-null, and $G_1,G_3 \in \Upsilon$. We use the notation of the triple theorem* (Theorem 1.3.8 a)) *associated to the $\mathfrak{M}_E$-triple*

$$(F \otimes G_1, F \otimes G_2, F \otimes G_3)$$

(Proposition 1.4.9)*, put* $\varphi := \varphi_{1,2}/(F \otimes G_3)$ (as in Proposition 1.3.7 a))*, and denote by*

$$\Psi_{F,i} : K_i\left(F \otimes G_1\right) \times K_{i+1}\left(F \otimes G_3\right) \longrightarrow K_i\left(F \otimes (G_1/G_3)\right),$$

$$(a,b) \longmapsto K_i\left(\psi_{1,3}\right) a + \Phi_i b$$

the corresponding group isomorphism (Theorem 1.3.8 a_4), Proposition 1.4.9). *Then*

$$G_1/G_3 \in \Upsilon, \qquad p(G_1/G_3) = p(G_1) + q(G_3), \qquad q(G_1/G_3) = q(G_1) + p(G_3),$$

$$\Phi_{i,(G_1/G_3),F} = \Psi_{F,i} \circ \left(\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}\right).$$

Since $G_1, G_3 \in \Upsilon$, the map

$$\Psi_{F,i} \circ \left(\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}\right) : \left(K_i\left(F\right)^{p(G_1)} \times K_{i+1}\left(F\right)^{q(G_1)}\right) \times$$

$$\times \left(K_{i+1}\left(F\right)^{p(G_3)} \times K_i\left(F\right)^{q(G_3)}\right) \longrightarrow K_i\left(F \otimes (G_1/G_3)\right)$$

is a group isomorphism. We put

$$\bar{p}(G_1/G_3) := p(G_1) + Q(G_3), \qquad \bar{q}(G_1/G_3) := q(G_1) + p(G_3),$$

$$\bar{\Phi}_{i,G_1/G_3,F} := \Psi_{F,i} \circ \left(\Phi_{i,G_1,F} \times \Phi_{(i+1),G_3,F}\right).$$

Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. We mark with a prime the above notation associated to F'. By the commutativity of the index maps (Axiom 1.2.8),

$$K_{i+1}\left(\phi \otimes id_{G_3}\right) \circ \delta_{2,3,i} = \delta'_{2,3,i} \circ K_i\left(\phi \otimes id_{(G_2/G_3)}\right).$$

Moreover

$$K_i\left(\phi \otimes id_{(G_1/G_3)}\right) \circ K_i\left(\varphi\right) = K_i\left(\varphi'\right) \circ K_i\left(\phi \otimes id_{(G_2/G_3)}\right),$$

$$K_i\left(\phi \otimes id_{(G_1/G_3)}\right) \circ K_i\left(\psi_{1,3}\right) = K_i\left(\psi'_{1,3}\right) \circ K_i\left(\phi \otimes id_{G_1}\right).$$

It follows

$$K_i\left(\phi \otimes id_{(G_1/G_3)}\right) \circ \Phi_i = K_i\left(\phi \otimes id_{(G_1/G_3)}\right) \circ K_i\left(\varphi\right) \circ \left(\delta_{2,3,i}\right)^{-1} =$$

$$= K_i\left(\varphi'\right) \circ K_i\left(\phi \otimes id_{(G_2/G_3)}\right) \circ \left(\delta_{2,3,i}\right)^{-1} =$$

$$= K_i\left(\varphi'\right) \circ \left(\delta'_{2,3,i}\right)^{-1} \circ K_{i+1}\left(\phi \otimes id_{G_3}\right) = \Phi'_i \circ K_{i+1}\left(\phi \otimes id_{G_3}\right).$$

We want to prove that the diagram

$$\begin{array}{ccc} K_i(F\otimes G_1)\times K_{i+1}(F\otimes G_3) & \xrightarrow{\Psi_{F,i}} & K_i(F\otimes(G_1/G_3)) \\ {\scriptstyle K_i(\phi\otimes id_{G_1})\times K_{i+1}(\phi\otimes id_{G_3})}\downarrow & & \downarrow{\scriptstyle K_i\left(\phi\otimes id_{(G_1/G_3)}\right)} \\ K_i(F'\otimes G_1)\times K_{i+1}(F'\otimes G_3) & \xrightarrow[\Psi_{F',i}]{} & K_i(F'\otimes(G_1/G_3)) \end{array}$$

is commutative. For $(a,b)\in K_i(F\otimes G_1)\times K_{i+1}(F\otimes G_3)$, by the above,

$$K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\Psi_{F,i}(a,b)=K_i\left(\phi\otimes id_{(G_1/G_3)}\right)(K_i(\psi_{1,3})a+\Phi_i b)=$$

$$=K_i\left(\phi\otimes id_{(G_1/G_3)}\right)K_i(\psi_{1,3})a+K_i\left(\phi\otimes id_{(G_1/G_3)}\right)\Phi_i b=$$

$$=K_i\left(\psi'_{1,3}\right)K_i(\phi\otimes id_{G_1})a+\Phi'_i K_{i+1}\left(\phi\otimes id_{G_3}\right)b=$$

$$=\Psi_{F',i}(K_i(\phi\otimes id_{G_1})a,K_{i+1}\left(\phi\otimes id_{G_3}\right)b)=$$

$$=\Psi_{F',i}(K_i(\phi\otimes id_{G_1})\times K_{i+1}\left(\phi\otimes id_{G_3}\right))(a,b)\,.$$

Thus the above diagram is commutative. It follows, since $G_1,G_3\in\Upsilon$, that the diagram

$$\begin{array}{ccc} K_i(F)^{p(G_1/G_3)}\times K_{i+1}(F)^{q(G_1/G_3)} & \xrightarrow{\bar{\Phi}_{i,(G_1/G_3),F}} & K_i(F\otimes(G_1/G_3)) \\ {\scriptstyle K_i(\phi)^{p(G_1/G_3)}\times K_{i+1}(\phi)^{q(G_1/G_3)}}\downarrow & & \downarrow{\scriptstyle K_i\left(\phi\otimes id_{(G_1/G_3)}\right)} \\ K_i(F')^{p(G_1/G_3)}\times K_{i+1}(F')^{q(G_1/G_3)} & \xrightarrow[\bar{\Phi}_{i,(G_1/G_3),F'}]{} & K_i(F'\otimes(G_1/G_3)) \end{array}$$

is commutative. Hence

$$G_1/G_3\in\Upsilon,\qquad p(G_1/G_3)=p(G_1)+q(G_3),\qquad q(G_1/G_3)=q(G_1)+p(G_3),$$

$$\Phi_{i,(G_1/G_3),F}=\Psi_{F,i}\circ(\Phi_{i,G_1,F}\times\Phi_{(i+1),G_3,F})\,.$$ ■

PROPOSITION 1.5.14 *Let* (G_1,G_2,G_3) *be an* $\mathfrak{M}_{\mathbf{C}}$*-triple such that* G_1/G_2 *and* G_1/G_3 *are nuclear,* G_1/G_3 *is* Υ*-null, and* $G_1,G_1/G_2\in\Upsilon$*. We use the notation of the triple theorem* (Theorem 1.3.8 b)) *associated to the* $\mathfrak{M}_E$*-triple*

$$(F\otimes G_1,F\otimes G_2,F\otimes G_3)$$

(Proposition 1.4.9)*, assume* ψ_{12} *K-null for all E-C*-algebras F, and denote by*

$$\Psi_{F,i}:K_i(F\otimes G_1)\times K_{i+1}(F\otimes(G_1/G_2))\longrightarrow K_i(F\otimes G_2)\,,$$

$$(a,b) \longmapsto \Phi_i' a + \delta_{1,2,(i+1)} b$$

the corresponding group isomorphism (Theorem 1.3.8 b_4), Proposition 1.4.9). *Then*

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + q(G_1/G_2), \qquad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) .$$

Since $G_1, G_1/G_2 \in \Upsilon$, the map

$$\Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) : \Big(K_i(F)^{p(G_1)} \times K_{i+1}(F)^{q(G_1)} \Big) \times$$
$$\times \Big(K_{i+1}(F)^{p(G_1/G_2)} \times K_i(F)^{q(G_1/G_2)} \Big) \longrightarrow K_i(F \otimes G_2)$$

is a group isomorphism. We put

$$\tilde{p}(G_2) := p(G_1) + q(G_1/G_2), \qquad \tilde{q}(G_2) := q(G_1) + p(G_1/G_2),$$

$$\tilde{\Phi}_{i,G_2,F} := \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) .$$

Let $F \xrightarrow{\phi} \bar{F}$ be a morphism in $\mathfrak{M}_E$. We mark with a bar the above notation associated to $\bar{F}$. By the commutativity of the index maps (Axiom 1.2.8),

$$K_i(\phi \otimes id_{G_2}) \circ \delta_{1,2,(i+1)} = \bar{\delta}_{1,2,(i+1)} \circ K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) .$$

Moreover

$$K_i(\phi \otimes id_{G_1}) \circ K_i(\varphi_{1,3}) = K_i(\bar{\varphi}_{1,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) ,$$
$$K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) = K_i(\bar{\varphi}_{2,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) .$$

It follows

$$K_i(\phi \otimes id_{G_2}) \circ \Phi_i' = K_i(\phi \otimes id_{G_2}) \circ K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1} =$$
$$= K_i(\bar{\varphi}_{2,3}) \circ K_i\left(\phi \otimes id_{G_3}\right) \circ K_i(\varphi_{1,3})^{-1} =$$
$$= K_i(\bar{\varphi}_{2,3}) \circ K_i(\bar{\varphi}_{1,3})^{-1} \circ K_i(\phi \otimes id_{G_1}) = \bar{\Phi}_i' \circ K_i(\phi \otimes id_{G_1}) .$$

We want to prove that the diagram

$$\begin{array}{ccc} K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2)) & \xrightarrow{\Psi_{F,i}} & K_i(F \otimes G_2) \\ {\scriptstyle K_i(\phi \otimes id_{G_1}) \times K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right)} \Big\downarrow & & \Big\downarrow {\scriptstyle K_i(\phi \otimes id_{G_2})} \\ K_i(\bar{F} \otimes G_1) \times K_{i+1}(\bar{F} \otimes (G_1/G_2)) & \xrightarrow[\Psi_{\bar{F},i}]{} & K_i(\bar{F} \otimes G_2) \end{array}$$

is commutative. For $(a,b) \in K_i(F \otimes G_1) \times K_{i+1}(F \otimes (G_1/G_2))$, by the above,

$$K_i(\phi \otimes id_{G_2})\Psi_{F,i}(a,b) = K_i(\phi \otimes id_{G_2})(\Phi'_i a + \delta_{1,2,(i+1)} b) =$$

$$= K_i(\phi \otimes id_{G_2})\Phi'_i a + K_i(\phi \otimes id_{G_2})\delta_{1,2,(i+1)} b =$$

$$= \bar{\Phi}'_i K_i(\phi \otimes id_{G_1}) a + \bar{\delta}_{1,2,(i+1)} K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) b =$$

$$= \Psi_{\bar{F},i}(K_i(\phi \otimes id_{G_1}) a, K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right) b) =$$

$$= \Psi_{\bar{F},i}\left(K_i(\phi \otimes id_{G_1}) \times K_{i+1}\left(\phi \otimes id_{(G_1/G_2)}\right)\right)(a,b) .$$

Thus the above diagram is commutative. Since $G_1, G_1/G_2 \in \Upsilon$, It follows that the diagram

$$\begin{array}{ccc} K_i(F)^{\tilde{p}(G_2)} \times K_{i+1}(F)^{\tilde{q}(G_2)} & \xrightarrow{\tilde{\Phi}_{i,G_2,F}} & K_i(F \otimes G_2) \\ {\scriptstyle K_i(\phi)^{p(G_2)} \times K_{i+1}(\phi)^{q(G_2)}} \downarrow & & \downarrow {\scriptstyle K_i(\phi \otimes id_{G_2})} \\ K_i(\bar{F})^{\tilde{p}(G_2)} \times K_{i+1}(\bar{F})^{\tilde{q}(G_2)} & \xrightarrow[\tilde{\Phi}_{i,G_2,\bar{F}}]{} & K_i(\bar{F} \otimes G_2) \end{array}$$

is commutative. Hence

$$G_2 \in \Upsilon, \qquad p(G_2) = p(G_1) + q(G_1/G_2), \qquad q(G_2) = q(G_1) + p(G_1/G_2),$$

$$\Phi_{i,G_2,F} = \Psi_{F,i} \circ (\Phi_{i,G_1,F} \times \Phi_{(i+1),(G_1/G_2),F}) . \qquad \blacksquare$$

PROPOSITION 1.5.15 *Let*

$$0 \longrightarrow G \xrightarrow{\varphi} H \xrightarrow{\psi} \mathbf{C} \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_{\mathbf{C}}$ *with G nuclear and H* Υ*-null and let* δ_i^F *denote the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0 .$$

Then

$$G \in \Upsilon, \qquad p(G) = 0, \qquad q(G) = 1, \qquad \Phi_{i,G,F=\delta_{i+1}^F} ,$$

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_\Upsilon .$$

By Proposition 1.5.6 b) and Proposition 1.5.4 b),

$$G \in \Upsilon, \qquad p(G) = 0, \qquad q(G) = 1, \qquad \Phi_{i,G,F=\delta^F_{i+1}} .$$

Since the diagram

$$\begin{array}{ccc} K_{i+1}(F) & \xrightarrow{\delta^F_{i+1}} & K_i(F \otimes G) \\ {\scriptstyle \Phi_{i,G,F}=\delta^F_{i+1}} \downarrow & & \downarrow {\scriptstyle \Phi_{(i+1),G,(F\otimes G)}=\delta^F_{G,i}} \\ K_i(F \otimes G) & \xrightarrow[\delta^F_{G,i}]{} & K_{i+1}((F \otimes G) \otimes G) \end{array}$$

is obviously commutative,

$$(0 \longrightarrow F \otimes G \xrightarrow{id_F \otimes \varphi} F \otimes H \xrightarrow{id_F \otimes \psi} F \longrightarrow 0) \in G_\Upsilon .$$

■

1.6 The Class Υ_1

Throughout this section F denotes an E-C*-algebra.

DEFINITION 1.6.1 *We denote by Υ_1 the class of unital C*-algebras G belonging to Υ such that*

$$p(G) = 1, \qquad q(G) = 0, \qquad \Phi_{i,G,F} = K_i(\phi_{G,F}),$$

where

$$\phi_{G,F} : F \longrightarrow F \otimes G, \quad x \longmapsto x \otimes 1_G .$$

PROPOSITION 1.6.2 $\mathbb{C} \in \Upsilon_1$.

In fact

$$\phi_{\mathbb{C},F} : F \longrightarrow F \otimes \mathbb{C}, \quad x \longmapsto x \otimes 1_{\mathbb{C}}$$

is an isomorphism. ■

PROPOSITION 1.6.3 *Let $G \in \Upsilon_1$ and let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. If we identify $K_i(F)$ with $K_i(F \otimes G)$ for all E-C*-algebras F using the group isomorphisms $\Phi_{i,G,F}$ then $K_i(\phi \otimes id_G)$ is identified with $K_i(\phi)$.*

The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi)} & K_i(F') \\ \Phi_{i,G,F}\big\downarrow & & \big\downarrow\Phi_{i,G,F'} \\ K_i(F\otimes G) & \xrightarrow[K_i(\phi\otimes id_G)]{} & K_i(F'\otimes G) \end{array}$$ ■

PROPOSITION 1.6.4 *Let G,H be C*-algebras and $\varphi : G \longrightarrow H$ and $\psi : H \longrightarrow G$ be a homotopy such that φ and ψ are unital. If $G \in \Upsilon_1$ then $H \in \Upsilon_1$.*

By Proposition 1.5.4 c),

$$H \in \Upsilon, \qquad p(H) = 1, \qquad q(H) = 0,$$

$$\Phi_{i,H,F} = K_i\left(id_F \otimes \varphi\right) \circ \Phi_{i,G,F} = K_i\left(id_F \otimes \varphi\right) \circ K_i\left(\phi_{G,F}\right) = K_i\left(\phi_{H,F}\right) .$$ ■

PROPOSITION 1.6.5 *If $(G_j)_{j\in J}$ is a finite family in Υ_1, $J \neq \emptyset$, then $\bigotimes_{j\in J} G_j \in \Upsilon_1$.*

$\bigotimes_{j\in J} G_j$ is unital and by Proposition 1.5.12 a), $\bigotimes_{j\in J} G_j \in \Upsilon$. Assume $J = \{1,2\}$ and let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccc} K_i(F) & \xrightarrow{K_i\left(\phi_{G_1,F}\right)} & K_i(F\otimes G_1) & \xrightarrow{K_i\left(\phi_{G_2,(F\otimes G_1)}\right)} & K_i(F\otimes G_1\otimes G_2) \\ K_i(\phi)\big\downarrow & & \big\downarrow K_i\left(\phi\otimes id_{G_1}\right) & & \big\downarrow K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right) \\ K_i(F') & \xrightarrow[K_i\left(\phi_{G_1,F'}\right)]{} & K_i(F'\otimes G_1) & \xrightarrow[K_i\left(\phi_{G_2,(F'\otimes G_1)}\right)]{} & K_i(F'\otimes G_1\otimes G_2) \end{array}$$

is commutative. Since

$$\phi_{(G_1\otimes G_2),F} = \phi_{G_2,(F\otimes G_1)} \circ \phi_{G_1,F}\,, \qquad \phi_{(G_1\otimes G_2),F'} = \phi_{G_2,(F'\otimes G_1)} \circ \phi_{G_1,F'}\,,$$

the diagram

$$\begin{array}{ccc} K_i(F) & \xrightarrow{K_i\left(\phi_{(G_1\otimes G_2),F}\right)} & K_i(F\otimes G_1\otimes G_2) \\ K_i(\phi)\big\downarrow & & \big\downarrow K_i\left(\phi\otimes id_{(G_1\otimes G_2)}\right) \\ K_i(F') & \xrightarrow[K_i\left(\phi_{(G_1\otimes G_2),F'}\right)]{} & K_i(F'\otimes G_1\otimes G_2) \end{array}$$

is commutative, which proves the assertion in this case. The general case follows now by induction with respect to *Card J*. ■

PROPOSITION 1.6.6 *If $G \in \Upsilon_1$ is nuclear then every exact sequence in $\mathfrak{M}_E$ belongs to G_Υ.*

Let

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & F_2 & \xrightarrow{\phi_2} & F_3 & \longrightarrow & 0 \\ & & {\scriptstyle \phi_{G,F_1}}\downarrow & & {\scriptstyle \phi_{G,F_2}}\downarrow & & \downarrow{\scriptstyle \phi_{G,F_3}} & & \\ 0 & \longrightarrow & F_1 \otimes G & \xrightarrow[\phi_1 \otimes id_G]{} & F_2 \otimes G & \xrightarrow[\phi_2 \otimes id_G]{} & F_3 \otimes G & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8) the diagram

$$\begin{array}{ccc} K_i(F_3) & \xrightarrow{\delta_i} & K_{i+1}(F_1) \\ {\scriptstyle \Phi_{i,G,F_3}=K_i(\phi_{G,F_3})}\downarrow & & \downarrow{\scriptstyle \Phi_{(i+1),G,F_1}=K_{i+1}(\phi_{G,F_1})} \\ K_i(F_3 \otimes G) & \xrightarrow[\delta_{G,i}]{} & K_{i+1}(F_1 \otimes G) \end{array}$$

is commutative, where δ_i denotes the index maps of the exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0 \,.$$ ■

PROPOSITION 1.6.7 *Let G be a C*-algebra.*

a) $\phi_{\tilde{G},F} = (id_F \otimes \lambda_G) \circ \phi_{\mathbf{C},F}$.

b) G is Υ-null iff $\tilde{G} \in \Upsilon_1$.

c) If G is Υ-null and $\varphi : G \longrightarrow G'$, $\psi : G \longrightarrow G'$ are C-homomorphisms then $K_i\,(id_F \otimes \tilde{\varphi}) = K_i\,(id_F \otimes \tilde{\psi})$. In particular if $G = G'$ then*

$$K_i\,(id_F \otimes \tilde{\varphi}) = id_{K_i\left(F \otimes \tilde{G}\right)} \approx id_{K_i(F)} \,.$$

a) is easy to see.

b) By Corollary 1.4.5 b), the sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(id_F \otimes \iota_G)} K_i\left(F \otimes \tilde{G}\right) \underset{K_i(id_F \otimes \lambda_G)}{\overset{K_i(id_F \otimes \pi_G)}{\rightleftarrows}} K_i(F) \longrightarrow 0$$

is split exact. By a) and Proposition 1.6.2,

$$K_i\left(\phi_{\tilde{G},F}\right) = K_i\left(id_F \otimes \lambda_G\right) \circ \Phi_{i,\mathbb{C},F} \ .$$

If $\tilde{G} \in \Upsilon_1$ then

$$\Phi_{i,\tilde{G},F} = K_i\left(\phi_{\tilde{G},F}\right) = K_i\left(id_F \otimes \lambda_G\right) \circ \Phi_{i,\mathbb{C},F} \ ,$$

so by Proposition 1.6.2, $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism, $K_i\left(id_F \otimes \iota_G\right) = 0$, $K_i\left(F \otimes G\right) = 0$, and G is Υ-null. If G is Υ-null then $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism so

$$K_i\left(\phi_{\tilde{G},F}\right) : K_i\left(F\right) \longrightarrow K_i\left(F \otimes G\right)$$

is an isomorphism and $\tilde{G} \in \Upsilon_1$.

c) Since $\tilde{\varphi} \circ \lambda_G = \tilde{\psi} \circ \lambda_G$,

$$K_i\left(id_F \otimes \tilde{\varphi}\right) \circ K_i\left(id_F \otimes \lambda_G\right) = K_i\left(id_F \otimes \tilde{\psi}\right) \circ K_i\left(id_F \otimes \lambda_G\right) \ .$$

By b), $\tilde{G} \in \Upsilon_1$ and so $K_i\left(id_F \otimes \lambda_G\right)$ is an isomorphism. Thus $K_i\left(id_F \otimes \tilde{\varphi}\right) = K_i\left(id_F \otimes \tilde{\psi}\right)$. ■

COROLLARY 1.6.8 *If $(G_j)_{j \in J}$ is a finite family of Υ-null C*-algebras and $G := \prod_{j \in J} G_j$ then $\tilde{G} \in \Upsilon_1$.*

By Proposition 1.5.9 a), G is Υ-null and by Proposition 1.6.7 b), $\tilde{G} \in \Upsilon_1$. ■

PROPOSITION 1.6.9 *Let*

$$0 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_\mathbb{C}$ such that G_1 is Υ-null, G_3 is nuclear, and G_2, G_3 are unital. Then $G_2 \in \Upsilon_1$ iff $G_3 \in \Upsilon_1$.

Since G_2 and G_3 are unital and ψ is surjective, $\psi(1_{G_2}) = 1_{G_3}$. It follows

$$\phi_{G_3,F} = (id_F \otimes \psi) \circ \phi_{G_2,F}\,, \qquad K_i\left(\phi_{G_3,F}\right) = K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right)\,.$$

By Proposition 1.5.6 a), $K_i\left(id_F \otimes \psi\right)$ is a group isomorphism,

$$G_2, G_3 \in \Upsilon\,, \qquad p(G_2) = p(G_3) = 1\,, \qquad q(G_2) = q(G_3) = 0\,,$$

$$\Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ \Phi_{i,G_2,F}\,.$$

If $G_2 \in \Upsilon_1$ then by the above,

$$\Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right) = K_i\left(\phi_{G_3,F}\right)\,,$$

so $G_3 \in \Upsilon_1$. If $G_3 \in \Upsilon_1$ then by the above,

$$K_i\left(id_F \otimes \psi\right) \circ K_i\left(\phi_{G_2,F}\right) = K_i\left(\phi_{G_3,F}\right) = \Phi_{i,G_3,F} = K_i\left(id_F \otimes \psi\right) \circ \Phi_{i,G_2,F}\,,$$

so $\Phi_{i,G_2,F} = K_i\left(\phi_{G_2,F}\right)$ and $G_2 \in \Upsilon_1$. ■

Chapter 2

Locally Compact Spaces

2.1 Tietze's Theorem

DEFINITION 2.1.1 *Let Ω be a topological space and F an E-C*-algebra. We endow canonically the C*-algebra $\mathscr{C}(\Omega,F)$ with the structure of an E-C*-algebra by putting*

$$\alpha x:\Omega \longrightarrow F\,,\quad \omega \longmapsto \alpha x(\omega)$$

for all $(\alpha,x)\in E\times F$. If Ω is a locally compact space then we endow $\mathscr{C}_0(\Omega,F)$ with the structure of on $E{-}C*{-}algebra$ in a similar way. If Ω' is an open set of a locally compact space Ω then we identify $\mathscr{C}_0(\Omega',F)$ with the E-ideal $\{\, x\in\mathscr{C}_0(\Omega,F)\mid x|(\Omega\setminus\Omega')=0\,\}$ of $\mathscr{C}_0(\Omega,F)$.

DEFINITION 2.1.2 *Let Ω be a locally compact space with $\mathscr{C}_0(\Omega,\mathbb{C})\in\Upsilon$. We put*

$$\Omega\in\Upsilon\,,\qquad p(\Omega):=p(\mathscr{C}_0(\Omega,\mathbb{C}))\,,\qquad q(\Omega):=q(\mathscr{C}_0(\Omega,\mathbb{C}))\,,$$

$$\Phi_{i,\Omega,F}:=\Phi_{i,\mathscr{C}_0(\Omega,\mathbb{C}),F}\,,\quad \Omega_\Upsilon:=\mathscr{C}_0(\Omega,\mathbb{C})_\Upsilon\,,\quad \Omega\in\Upsilon_1:\Longleftrightarrow\mathscr{C}_0(\Omega,\mathbb{C})\in\Upsilon_1\,.$$

We say that Ω is **Υ-null** *if $\mathscr{C}_0(\Omega,\mathbb{C})$ is Υ-null. We say that Ω is* **null-homotopic** *if $\mathscr{C}_0(\Omega,\mathbb{C})$ is null-homotopic.*

PROPOSITION 2.1.3 *If Ω is a locally compact space and if Ω^* denotes its Alexandroff compactification then Ω is Υ-null iff $\Omega^*\in\Upsilon_1$.*

The Proposition is a particular case of Proposition 1.6.7. ■

LEMMA 2.1.4 *Let Ω be a locally compact space.*

a) $\mathscr{C}_0(\Omega,\mathbb{C})$ is nuclear.

b) $\mathscr{C}_0(\Omega,F)\approx F\otimes\mathscr{C}_0(\Omega,\mathbb{C})$.

c) If Ω is a finite compact space then $\Omega\in\Upsilon$, $p(\Omega)=Card\,\Omega$, $q(\Omega)=0$, and every exact sequence in $\mathfrak{M}_E$ belongs to Ω_Υ.

a) [5] Theorem T.6.20.

b) [5] Proposition T.5.11,

c) follows from Proposition 1.5.9 c). ■

COROLLARY 2.1.5 (Tietze's Theorem) *Let* Ω *be a locally compact space,* Γ *a closed set of* Ω*,* $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ *the inclusion map, and*

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Gamma, F)\,, \quad x \longmapsto x|\Gamma\,.$$

Then

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$.

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.9. ■

COROLLARY 2.1.6 *If*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$ *and* Ω *a locally compact space then*

$$0 \longrightarrow \mathscr{C}_0(\Omega, F_1) \xrightarrow{\phi_1 \otimes id_G} \mathscr{C}_0(\Omega, F_2) \xrightarrow{\phi_2 \otimes id_G} \mathscr{C}_0(\Omega, F_3) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$.

By Lemma 2.1.4 a),b), the assertion follows from Proposition 1.4.10. ■

PROPOSITION 2.1.7 *Let*

$$0 \longrightarrow F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} F_3 \longrightarrow 0$$

be an exact sequence in $\mathfrak{M}_E$*,* Ω *a locally compact space,* Γ *a closed set of* Ω*,* $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, \mathbb{C}) \longrightarrow \mathscr{C}_0(\Omega, \mathbb{C})$ *the inclusion map, and*

$$\psi : \mathscr{C}_0(\Omega, \mathbb{C}) \longrightarrow \mathscr{C}_0(\Gamma, \mathbb{C})\,, \quad x \longmapsto x|\Gamma\,.$$

a) $G := \{\, x \in \mathscr{C}_0\,(\Omega, F_2) \mid x|\Gamma \in \mathscr{C}_0\,(\Gamma, F_1)\,\}$ *is a closed E-ideal of* $\mathscr{C}_0\,(\Omega, F_2)$*; we denote by* $\varphi' : G \longrightarrow \mathscr{C}_0\,(\Omega, F_2)$ *the inclusion map.*

b) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow G \xrightarrow{\varphi'} \mathscr{C}_0\,(\Omega, F_2) \xrightarrow{\phi_2 \otimes \psi} \mathscr{C}_0\,(\Gamma, F_3) \longrightarrow 0$$

is exact.

a) is easy to see.

b) We put

$$G_1 := \mathscr{C}_0\,(\Omega \setminus \Gamma, \mathbf{C})\,, \qquad G_2 := \mathscr{C}_0\,(\Omega, \mathbf{C})\,, \qquad G_3 := \mathscr{C}_0\,(\Gamma, \mathbf{C})\;.$$

Let us consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_1 & \xrightarrow{\phi_1 \otimes id_{G_1}} & F_2 \otimes G_1 & \xrightarrow{\phi_2 \otimes id_{G_1}} & F_3 \otimes G_1 & \longrightarrow & 0 \\
 & & {\scriptstyle id_{F_1} \otimes \varphi}\downarrow & & {\scriptstyle id_{F_2} \otimes \varphi}\downarrow & & {\scriptstyle id_{F_3} \otimes \varphi}\downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_2 & \xrightarrow{\phi_1 \otimes id_{G_2}} & F_2 \otimes G_2 & \xrightarrow{\phi_2 \otimes id_{G_2}} & F_3 \otimes G_2 & \longrightarrow & 0 \\
 & & {\scriptstyle id_{F_1} \otimes \psi}\downarrow & & {\scriptstyle id_{F_2} \otimes \psi}\downarrow & & {\scriptstyle id_{F_3} \otimes \psi}\downarrow & & \\
0 & \longrightarrow & F_1 \otimes G_3 & \xrightarrow{\phi_1 \otimes id_{G_3}} & F_2 \otimes G_3 & \xrightarrow{\phi_2 \otimes id_{G_3}} & F_3 \otimes G_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.1.4 a), Proposition 1.4.9, and Proposition 1.4.10, its columns and rows are exact. It follows that $\phi_2 \otimes \psi$ is surjective. Let $x \in Ker\,(\phi_2 \otimes \psi)$. Then

$$(id_{F_3} \otimes \psi)(\phi_2 \otimes id_{G_2})x = (\phi_2 \otimes \psi)x = 0\,,$$

so there is a $y \in F_2 \otimes G_1$ with

$$(\phi_2 \otimes \varphi)y = (id_{F_3} \otimes \varphi)(\phi_2 \otimes id_{G_1})y = (\phi_2 \otimes id_{G_2})x\;.$$

Then

$$(\phi_2 \otimes id_{G_2})(x - (id_{F_2} \otimes \varphi)y) = (\phi_2 \otimes id_{G_2})x - (\phi_2 \otimes \varphi)y = 0\,,$$

so there is a $z \in F_1 \otimes G_2$ with

$$(\phi_1 \otimes id_{G_2})z = x - (id_{F_2} \otimes \varphi)y\,.$$

Thus

$$x = (id_{F_2} \otimes \varphi)y + (\phi_1 \otimes id_{G_2})z \in G\,, \qquad Ker\,(\phi_2 \otimes \psi) \subset G\,.$$

Let now $x \in G$. By Proposition 1.4.9, there is a $y \in \mathscr{C}_0\,(\Omega, F_1) = F_1 \otimes G_2$ with $x|\Gamma = y|\Gamma$. There is a $z \in \mathscr{C}_0\,(\Omega \setminus \Gamma, F_2) = F_2 \otimes G_1$ with

$$(id_{F_2} \otimes \varphi)z = x - (\phi_1 \otimes id_{G_2})y\,.$$

We get

$$(\phi_2 \otimes \psi)x = (\phi_2 \otimes \psi)(\phi_1 \otimes id_{G_2})y + (\phi_2 \otimes \psi)(id_{F_2} \otimes \varphi)z =$$

$$= ((\phi_2 \circ \phi_1) \otimes \psi)y + (\phi_2 \otimes (\psi \circ \varphi))z = 0\,,$$

$G \subset Ker\,(\phi_2 \otimes \psi)$. ■

Remark. If we put $F_1 := 0$ and $F_2 = F_3$ in the above Proposition then we obtain Tietze's Theorem (Corollary 2.1.5).

PROPOSITION 2.1.8 (Topological six-term sequence) *Let Ω be a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0\,(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0\,(\Omega, F)$ the inclusion map,*

$$\psi : \mathscr{C}_0\,(\Omega, F) \longrightarrow \mathscr{C}_0\,(\Gamma, F)\,, \quad x \longmapsto x|\Gamma\,,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$ (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathscr{C}_0\,(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0\,(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0\,(\Gamma, F) \longrightarrow 0\,.$$

a) Assume $\Omega \setminus \Gamma$ is Υ-null.

a_1) $K_i(\psi) : K_i(\mathscr{C}_0\,(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0\,(\Gamma, F))$ is a group isomorphism.

a_2) If $\Omega \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega, \Gamma \in \Upsilon, \qquad p(\Omega) = p(\Gamma), \qquad q(\Omega) = q(\Gamma),$$

$$\Phi_{i,\Gamma,F} = K_i\,(id_F \otimes \psi) \circ \Phi_{i,\Omega,F}, \qquad \Omega_\Upsilon = \Gamma_\Upsilon .$$

b) Assume Ω is Υ-null.

b_1) $\delta_i : K_i(\mathscr{C}_0\,(\Gamma, F)) \longrightarrow K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma, F))$ is a group isomorphism.

b_2) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Gamma \in \Upsilon, \qquad p(\Omega \setminus \Gamma) = q(\Gamma), \qquad q(\Omega \setminus \Gamma) = p(\Gamma),$$

$$\Phi_{i,\Gamma,F} = \Phi_{(i+1),(\Omega\setminus\Gamma),F} \circ \delta_i .$$

c) Assume Γ is Υ-null.

c_1) $K_i(\varphi) : K_i(\mathscr{C}_0\,(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}_0\,(\Omega, F))$ is a group isomorphism.

c_2) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Omega \in \Upsilon$ then

$$\Omega \setminus \Gamma, \Omega \in \Upsilon, \qquad p(\Omega \setminus \Gamma) = p(\Omega), \qquad q(\Omega \setminus \Gamma) = q(\Omega),$$

$$\Phi_{i,\Omega,F} = K_i\,(id_F \otimes \varphi) \circ \Phi_{i,(\Omega\setminus\Gamma),F}, \qquad (\Omega \setminus \Gamma)_\Upsilon = \Omega_\Upsilon .$$

The assertions follow from Lemma 2.1.4 a),b) and Proposition 1.5.6. ■

COROLLARY 2.1.9 *Let Ω be a locally compact space, $\omega \in \Omega$ such that $\Omega \setminus \{\omega\}$ is Υ-null, Γ a closed set of Ω,*

$$\Omega' := (\Omega \setminus \{\omega\}) \setminus \Gamma, \qquad \Gamma' := \Gamma \setminus \{\omega\},$$

$\varphi : \mathscr{C}_0\,(\Omega', F) \longrightarrow \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F)$ the inclusion map,

$$\psi : \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}_0\left(\Gamma', F\right), \quad x \longmapsto x|\Gamma',$$

and δ_i the index maps of the exact sequence in $\mathfrak{M}_E$ (Tietze's Theorem (Corollary 2.1.5))

$$0 \longrightarrow \mathscr{C}_0\left(\Omega', F\right) \xrightarrow{\varphi} \mathscr{C}_0\,(\Omega \setminus \{\omega\}, F) \xrightarrow{\psi} \mathscr{C}_0\left(\Gamma', F\right) \longrightarrow 0 .$$

a) $\delta_i : K_i\,(\mathscr{C}_0\,(\Gamma', F)) \longrightarrow K_{i+1}\,(\mathscr{C}_0\,(\Omega', F))$ is a group isomorphism.

b) If $\Omega' \in \Upsilon$ *or* $\Gamma' \in \Upsilon$ *then*

$$\Omega', \Gamma' \in \Upsilon, \qquad p(\Omega') = q(\Gamma'), \qquad q(\Omega') = p(\Gamma'),$$

$$\Phi_{i,\Gamma',F} = \Phi_{(i+1),\Omega',F} \circ \delta_i .$$

c) If Γ *is finite then*

$$\Omega' \in \Upsilon, \qquad p(\Omega') = 0, \qquad q(\Omega') = Card\,\Gamma' .$$

a) and b) follow from the Topological six-term sequence (Proposition 2.1.8 b)).

c) follows from b) and Lemma 2.1.4 c). ■

COROLLARY 2.1.10 *Let* Ω, Ω' *be locally compact spaces,* $\omega \in \Omega$*, and* $\omega' \in \Omega'$ *such that* $\Omega' \setminus \{\omega'\}$ *is null-homotopic.*

a) $K_i(\mathscr{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)) \approx K_i(\mathscr{C}_0((\Omega \setminus \{\omega\}) \times \Omega', F))$.

b) If also $\Omega \setminus \{\omega\}$ *is null-homotopic then* $\mathscr{C}_0((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F)$ *is K-null.*

a) The sequence in $\mathfrak{M}_E$ (with obvious notation)

$$0 \longrightarrow \mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right) \xrightarrow{\varphi} \mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right)$$

$$\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right) \xrightarrow{\psi} \mathscr{C}_0\left(\{\omega\} \times (\Omega' \setminus \{\omega'\}), F\right) \longrightarrow 0$$

is exact and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c_1)).

b) By Proposition 1.4.2 c) and Lemma 2.1.4 b), $(\Omega \setminus \{\omega\}) \times \Omega'$ is null-homotopic and so K-null (Proposition 1.5.4 a)). By a),

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{(\omega, \omega')\}, F\right)\right)$$

is K-null. ■

PROPOSITION 2.1.11 (Topological triple) *Let Ω_1 be a locally compact space, Ω_2 an open set of Ω_1, Ω_3 an open set of Ω_2, and $\varphi : \mathscr{C}_0(\Omega_2 \setminus \Omega_3, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ the inclusion map. For all $j,k \in \{1,2,3\}$, $j < k$, put*

$$\psi_{j,k} : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega_j \setminus \Omega_k, F)\,, \quad x \longmapsto x|(\Omega_j \setminus \Omega_k)$$

and denote by $\varphi_{j,k} : \mathscr{C}_0(\Omega_k, F) \longrightarrow \mathscr{C}_0(\Omega_j, F)$ the inclusion map and by $\delta_{j,k,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega_k, F) \xrightarrow{\varphi_{j,k}} \mathscr{C}_0(\Omega_j, F) \xrightarrow{\psi_{j,k}} \mathscr{C}_0(\Omega_j \setminus \Omega_k, F) \longrightarrow 0\,.$$

a) Assume $\mathscr{C}_0(\Omega_2, F)$ K-null.

a$_1$) $\delta_{2,3,i} : K_i(\mathscr{C}_0(\Omega_2 \setminus \Omega_3, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_3, F))$ is a group isomorphism.

a$_2$) $\delta_{2,3,i} = \delta_{1,3,i} \circ K_i(\varphi)$.

a$_3$) $\varphi_{1,3}$ is K-null.

a$_4$) If we put $\Phi_i := K_i(\varphi) \circ (\delta_{2,3,i})^{-1}$ then

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega_1, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(\mathscr{C}_0(\Omega_3, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}_0(\Omega_1, F)) \times K_{i+1}(\mathscr{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F))\,,$$

$$(a,b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b$$

is a group isomorphism.

a$_5$) If Ω_2 is Υ-null and $\Omega_1, \Omega_3 \in \Upsilon$ then

$$\Omega_1 \setminus \Omega_3 \in \Upsilon,\ p(\Omega_1 \setminus \Omega_3) = p(\Omega_1) + q(\Omega_3)\,,\ q(\Omega_1 \setminus \Omega_3) = q(\Omega_1) + p(\Omega_3)\,,$$

and (with the notation of Proposition 1.5.13*)*

$$\Phi_{i,(\Omega_1 \setminus \Omega_3),F} = \Psi_{F,i} \circ (\Phi_{i,\Omega_1,F} \times \Phi_{(i+1),\Omega_3,F})\,.$$

b) Assume $\mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ K-null.

b$_1$) $\delta_{2,3,i} = 0$.

b$_2$) $K_i(\varphi_{1,3}) : K_i(\mathscr{C}_0(\Omega_3, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_1, F))$ *is a group isomorphism.*

b$_3$) *If we put* $\Phi_i := K_i(\varphi_{1,3})^{-1} \circ K_i(\varphi_{1,2})$ *then the map*

$$\Psi : K_i(\mathscr{C}_0(\Omega_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_3, F)) \times K_i(\mathscr{C}_0(\Omega_2 \setminus \Omega_3, F)),$$

$$b \longmapsto (\Phi_i b, K_i(\psi_{2,3})b)$$

is a group isomorphism.

b$_4$) *If* $\psi_{1,2}$ *is K-null and if we put* $\Phi'_i := K_i(\varphi_{2,3}) \circ K_i(\varphi_{1,3})^{-1}$ (by c_2)) *then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F)) \xrightarrow{\delta_{1,2,(i+1)}} K_i(\mathscr{C}_0(\Omega_2, F)) \underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}}$$

$$\underset{\Phi'_i}{\overset{K_i(\varphi_{1,2})}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega_1, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}_0(\Omega_1, F)) \times K_{i+1}(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_2, F)),$$

$$(a, b) \longmapsto \Phi'_i a + \delta_{1,2,(i+1)} b$$

is a group isomorphism.

b$_5$) *If* $\Omega_1 \setminus \Omega_3$ *is* Υ*-null,* $\Omega_1, \Omega_1 \setminus \Omega_2 \in \Upsilon$*, and* $\psi_{1,2}$ *is K-null then*

$$\Omega_2 \in \Upsilon,\ p(\Omega_2) = p(\Omega_1) + q(\Omega_1 \setminus \Omega_2),\ q(\Omega_2) = q(\Omega_1) + p(\Omega_1 \setminus \Omega_2).$$

c) *Assume* $\mathscr{C}_0(\Omega_1, F)$ *K-null and put*

$$\psi : \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_2, F), \quad x \longmapsto x|(\Omega_1 \setminus \Omega_2).$$

c$_1$) $\delta_{1,2,i}$ *and* $\delta_{1,3,i}$ *are group isomorphisms.*

c$_2$) $K_i(\varphi_{2,3}) \circ \delta_{1,3,(i+1)} = \delta_{1,2,(i+1)} \circ K_{i+1}(\psi)$.

c$_3$) *Let* $\varphi' : \mathscr{C}_0(\Omega_1 \setminus \Omega_2, F) \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Omega_3, F)$ *be a morphism in* $\mathfrak{M}_E$ *such that*

$$K_i(\psi \circ \varphi') = id_{K_i(\mathscr{C}_0(\Omega_1 \setminus \Omega_2, F))}.$$

If we put

$$\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$$

then $K_i(\varphi_{2,3})\circ\Phi_i = id_{K_i(\mathscr{C}_0(\Omega_2,F))}$. *If in addition* $\psi_{2,3}$ *is K-null then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega_2\setminus\Omega_3,F)) \xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathscr{C}_0(\Omega_3,F)) \underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega_2,F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}((\mathscr{C}_0(\Omega_2\setminus\Omega_3,F))\times K_i(\mathscr{C}_0(\Omega_2,F)) \longrightarrow K_i(\mathscr{C}_0(\Omega_3,F)),$$

$$(a,b)\longmapsto \delta_{2,3,(i+1)}a+\Phi_i b$$

is a group isomorphism.

Up to a_5) and b_5) the Proposition follows from Tietze's Theorem (Corollary 2.1.5) and from the triple theorem (Theorem 1.3.8) (and Lemma 2.1.4 a),b)). a_5) follows from Proposition 1.5.13 and b_5) follows from Proposition 1.5.14. ■

COROLLARY 2.1.12 *Let* $F\overset{\phi}{\to}F'$ *be a morphism in* $\mathfrak{M}_E$. *We use the notation and hypotheses of* Proposition 2.1.11 *and the hypothesis that* $\mathscr{C}_0(\Omega_2,F)$ *and* $\mathscr{C}_0(\Omega_2,F')$ *are K-null, and mark with an accent those notation associated to* F'. *We put for all* $j\in\{1,2,3\}$ *and for all* $j,k\in\{1,2,3\}$, $j<k$,

$$\phi_j:\mathscr{C}_0(\Omega_j,F)\longrightarrow\mathscr{C}_0\left(\Omega_j,F'\right),\quad x\longmapsto\phi\circ x,$$

$$\phi_{j,k}:\mathscr{C}_0(\Omega_j\setminus\Omega_k,F)\longrightarrow\mathscr{C}_0\left(\Omega_j\setminus\Omega_k,F'\right),\quad x\longmapsto\phi\circ x.$$

a) $\Phi'_i\circ K_{i+1}(\phi_3)=K_i(\phi_{1,3})\circ\Phi_i$.

b) If we identify $K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F))$ *with* $K_i(\mathscr{C}_0(\Omega_1,F))\times K_{i+1}(\mathscr{C}_0(\Omega_3,F))$ *and* $K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F'))$ *with* $K_i(\mathscr{C}_0(\Omega_1,F'))\times K_{i+1}(\mathscr{C}_0(\Omega_3,F'))$ *using the isomorphisms of* Proposition 2.1.11 a_4) *then*

$$K_i(\phi_{1,3}):K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_3,F))\longrightarrow K_i(\mathscr{C}_0\left(\Omega_1\setminus\Omega_3,F'\right)),$$

$$(a,b)\longrightarrow(K_i(\phi_1)a,K_{i+1}(\phi_3)b)$$

is a group isomorphism.

a) The diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathscr{C}_0(\Omega_3,F) & \xrightarrow{\varphi_{2,3}} & \mathscr{C}_0(\Omega_2,F) & \xrightarrow{\psi_{2,3}} & \mathscr{C}_0(\Omega_2\setminus\Omega_3,F)\longrightarrow 0 \\ \big\downarrow \phi_3 & & \big\downarrow \phi_2 & & \big\downarrow \phi_{2,3} \\ 0 \longrightarrow \mathscr{C}_0(\Omega_3,F')) & \xrightarrow[\varphi'_{2,3}]{} & \mathscr{C}_0(\Omega_2,F') & \xrightarrow[\psi'_{2,3}]{} & \mathscr{C}_0(\Omega_2\setminus\Omega_3,F')\longrightarrow 0 \end{array}$$

is obviously commutative and has exact rows. By the commutativity of the index maps (Axiom 1.2.8),

$$K_{i+1}(\phi_3)\circ\delta_{2,3,i} = (\delta_{2,3,i})'\circ K_i(\phi_{2,3}),$$

$$\left((\delta_{2,3,i})'\right)^{-1}\circ K_{i+1}(\phi_3) = K_i(\phi_{2,3})\circ(\delta_{2,3,i})^{-1} .$$

By the above, since $\phi_{1,3}\circ\varphi = \varphi'\circ\phi_{2,3}$,

$$K_i(\phi_{1,3})\circ\Phi_i = K_i(\phi_{1,3})\circ K_i(\varphi)\circ(\delta_{2,3,i})^{-1} = K_i(\varphi')\circ K_i(\phi_{2,3})\circ(\delta_{2,3,i})^{-1} =$$

$$= K_i(\varphi')\circ\left((\delta_{2,3,i})'\right)^{-1}\circ K_{i+1}(\phi_3) = \Phi'_i\circ K_{i+1}(\phi_3) .$$

b) follows from a) and Proposition 2.1.11 a_4). ■

2.2 Alexandroff Compactification

THEOREM 2.2.1 (Alexandroff K-theorem) *Let Ω be a locally compact space and Ω^* its Alexandroff compactification. We denote by*

$$\varphi : \mathscr{C}_0(\Omega,F)\longrightarrow \mathscr{C}(\Omega^*,F)$$

the inclusion map and put

$$\lambda : F\longrightarrow \mathscr{C}(\Omega^*,F), \quad y\longmapsto y1_{\mathscr{C}(\Omega^*,\mathbb{C})}$$

a) The map

$$K_i(\mathscr{C}_0(\Omega,F))\times K_i(F)\longrightarrow K_i(\mathscr{C}(\Omega^*,F)), \quad (a,b)\longmapsto K_i(\varphi)a + K_i(\lambda)b$$

is a group isomorphism.

b) If $\Omega \in \Upsilon$ *then*

$$\Omega^* \in \Upsilon\,, \qquad p(\Omega^*) = p(\Omega) + 1\,, \qquad q(\Omega^*) = q(\Omega) \qquad \Omega_\Upsilon \subset \Omega^*_\Upsilon\,.$$

c) Ω *is* Υ*-null iff* $\Omega^* \in \Upsilon_1$.

$\mathscr{C}(\Omega^*, \mathbf{C})$ is the unitization of $\mathscr{C}_0(\Omega, \mathbf{C})$.

a) Since

$$\mathscr{C}_0(\Omega, F) \approx F \otimes \mathscr{C}_0(\Omega, \mathbf{C})\,, \qquad \mathscr{C}_0(\Omega^*, F) \approx F \otimes \mathscr{C}_0(\Omega^*, \mathbf{C})$$

(Lemma 2.1.4 b)), the assertion follows from Corollary 1.4.5 b).

b) follows from Corollary 1.5.8.

c) follows from Proposition 1.6.7 b). ∎

COROLLARY 2.2.2 *Let* Ω_1 *and* Ω_2 *be locally compact spaces,* Ω_1^*, Ω_2^* *their Alexandroff compactification, respectively,* $\vartheta : \Omega_1 \longrightarrow \Omega_2$ *a proper continuous map,* $\vartheta^* : \Omega_1^* \longrightarrow \Omega_2^*$ *its continuous extension, and*

$$\phi : \mathscr{C}_0(\Omega_2, F) \longrightarrow \mathscr{C}_0(\Omega_1, F)\,, \quad x \longmapsto x \circ \vartheta\,,$$

$$\phi^* : \mathscr{C}(\Omega_2^*, F) \longrightarrow \mathscr{C}(\Omega_1^*, F)\,, \quad x \longmapsto x \circ \vartheta^*\,.$$

a) If we identify $K_i\left(\mathscr{C}\left(\Omega_j^*, F\right)\right)$ *with* $K_i(\mathscr{C}_0(\Omega_j, F)) \times K_i(F)$ *for every* $j \in \{1, 2\}$ *using the group isomorphisms of the Alexandroff K-theorem* (Theorem 2.2.1 a)) *then*

$$K_i(\phi^*) : K_i(\mathscr{C}(\Omega_2^*, F)) \longrightarrow K_i(\mathscr{C}(\Omega_1^*, F))\,, \quad (a, b) \longmapsto (K_i(\phi)a, b)\,.$$

b) Let $\vartheta' : \Omega_1 \longrightarrow \Omega_2$ *be a proper continuous map and let* ϕ', ϕ'^* *be the above maps associated to* ϑ'*. If* Ω_2 *is* Υ*-null then* $K_i(id_F \otimes \phi^*) = K_i(id_F \otimes \phi'^*)$*. In particular if* $\Omega_1 = \Omega_2$ *then*

$$K_i(id_F \otimes \phi^*) = id_{K_i(\mathscr{C}(\Omega_1^*, F))}\,.$$

a) follows from Corollary 1.4.5 c).

b) follows from Proposition 1.6.7 c). ■

COROLLARY 2.2.3 *Let $F \overset{\phi}{\to} F'$ be a morphism in $\mathfrak{M}_E$. We use the notation of the Alexandroff K-theorem* (Theorem 2.2.1) *and put*

$$\phi_\Omega : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega, F'\right), \quad x \longmapsto \phi \circ x,$$

$$\phi_{\Omega^*} : \mathscr{C}(\Omega^*, F) \longrightarrow \mathscr{C}\left(\Omega^*, F'\right), \quad x \longmapsto \phi \circ x .$$

If we identify $K_i(\mathscr{C}(\Omega^*, F))$ with $K_i(\mathscr{C}_0(\Omega, F)) \times K_i(F)$ and $K_i(\mathscr{C}(\Omega^*, F'))$ with $K_i(\mathscr{C}_0(\Omega, F')) \times K_i(F')$ using the group isomorphism of the Alexandroff K-theorem (Theorem 2.2.1 a)) *then*

$$K_i(\phi_{\Omega^*}) : K_i(\mathscr{C}(\Omega^*, F)) \longrightarrow K_i\left(\mathscr{C}\left(\Omega^*, F'\right)\right), \quad (a,b) \longmapsto (K_i(\phi_\Omega)a, K_i(\phi)b) .$$

The assertion follows from Corollary 1.4.5 c). ■

COROLLARY 2.2.4 *We use the notation of the Alexandroff K-theorem* (Theorem 2.2.1 a)) *and denote by ω_∞ the Alexandroff point of Ω. Let Ω' be a locally compact space,*

$$\varphi' : \mathscr{C}_0\left(\Omega \times \Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega^* \times \Omega', F\right)$$

the inclusion map, and

$$\lambda' : \mathscr{C}_0\left(\Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega^* \times \Omega', F\right), \quad x \longmapsto \tilde{x},$$

where

$$\tilde{x} : \Omega^* \times \Omega' \longrightarrow F, \quad (\omega, \omega') \longmapsto x(\omega') .$$

Then the map

$$K_i\left(\mathscr{C}_0\left(\Omega \times \Omega', F\right)\right) \times K_i(F) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega^* \times \Omega', F\right)\right),$$

$$(a,b) \longmapsto K_i(\varphi')a + K_i(\lambda')b$$

is a group isomorphism.

If we put

$$\psi' : \mathscr{C}_0\left(\Omega^* \times \Omega', F\right) \longrightarrow \mathscr{C}_0\left(\Omega', F\right), \quad x \longmapsto x(\omega_\infty, \cdot)$$

then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega \times \Omega', F\right) \xrightarrow{\varphi'} \mathscr{C}_0\left(\Omega^* \times \Omega', F\right) \underset{\lambda'}{\overset{\psi'}{\rightleftarrows}} \mathscr{C}_0\left(\Omega', F\right) \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ and the assertion follows from the split exact axiom (Axiom 1.2.3). ■

2.3 Topological Sums of Locally Compact Spaces

PROPOSITION 2.3.1 (Product Theorem) *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, Ω its topological sum, and for every $j \in J$ let $\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ be the inclusion map and*

$$\psi_j : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Omega_j, F), \quad x \longmapsto x|\Omega_j .$$

a)

$$\Phi_i : \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F)), \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} K_i(\varphi_j)a_j$$

is a group isomorphism and

$$\Psi_i : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j, F)), \quad a \longmapsto (K_i(\psi_j)a)_{j\in J}$$

is its inverse.

b) If all Ω_j, $j \in J$, belong to Υ then

$$\Omega \in \Upsilon, \qquad p(\Omega) = \sum_{j\in J} p(\Omega_j), \qquad q(\Omega) = \sum_{j\in J} q(\Omega_j),$$

$$\Phi_{i,\Omega,F} = \prod_{j\in J} \Phi_{i,\Omega_j,F}, \qquad \bigcap_{j\in J} (\Omega_j)_\Upsilon \subset \Omega_\Upsilon .$$

c) If Ω_j is Υ-null for every $j \in J$ then Ω is also Υ-null and $\Omega^* \in \Upsilon_1$, where Ω^* denotes the Alexandroff compactification of Ω.

a) follows from Proposition 1.3.3.

b) follows from Proposition 1.5.9.

c) By b), Ω is Υ-null and by Alexandroff's K-theorem (Theorem 2.2.1 a)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.3.2 *Let Ω be a locally compact space, Γ a closed set of Ω, and $(\Omega_j)_{j\in J}$ a finite family of pairwise disjoint open sets of Ω such that $\bigcup_{j\in J} \Omega_j = \Omega \setminus \Gamma$. We denote for every $j \in J$ by $\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map and assume that the maps*

$$K_i(\varphi_j) : K_i(\mathscr{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F))$$

are group isomorphisms. If $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ denotes the inclusion map and if we identify the above groups then $K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F))^J$ and

$$K_i(\varphi) : K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F)), \quad (a_j)_{j\in J} \longmapsto \sum_{j\in J} a_j \,. \qquad ■$$

COROLLARY 2.3.3 *Let Ω be a locally compact space such that $\mathscr{C}_0(\Omega, F)$ is K-null and Γ a closed set of Ω.*

a) $K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}(\Gamma, F))$.

b) Assume Γ finite and Ω Υ-null, put

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and denote by $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}(\Gamma, F) \longrightarrow 0 \,.$$

Then

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(F)^{\Gamma},$$

$$\Omega \setminus \Gamma \in \Upsilon, \qquad p(\Omega \setminus \Gamma) = 0, \qquad q(\Omega \setminus \Gamma) = Card\,\Gamma, \qquad \Phi_{i,(\Omega\setminus\Gamma),F} = \delta_{i+1} \,.$$

a) Since $\mathscr{C}_0(\Omega, F)$ is K-null, the assertion follows from the six-term axiom (Axiom 1.2.7).

b) follows from a), Lemma 2.1.4 c), and the Product Theorem (Proposition 2.3.1). ■

COROLLARY 2.3.4 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, Ω its topological sum, and Ω^* the Alexandroff compactification of Ω.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F))\,,\, K_i(\mathscr{C}(\Omega^*,F)) \approx K_i(F)\times\prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F))\,.$

b) If all Ω_j, $j\in J$, belong to Υ then

$$\Omega^*\in\Upsilon\,,\qquad p(\Omega^*)=1+\sum_{j\in J}p(\Omega_j)\,,\qquad q(\Omega^*)=\sum_{j\in J}q(\Omega_j)\,.$$

The assertion follows immediately from the Product Theorem (Proposition 2.3.1 a)) and the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 2.3.5 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces such that $\mathscr{C}_0(\Omega_j,F)$ is K-null for every $j\in J$ and let Γ_j be a closed set of Ω_j for every $j\in J$. We denote by Ω the Alexandroff compactification of the topological sum of the family $(\Omega_j\setminus\Gamma_j)_{j\in J}$.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times\prod_{j\in J} K_{i+1}(\mathscr{C}_0(\Gamma_j,F))$.

b) If for every $j\in J$, Ω_j is Υ-null and Γ_j is finite then

$$\Omega\in\Upsilon\,,\qquad p(\Omega)=1\,,\qquad q(\Omega)=\sum_{j\in J}\mathit{Card}\,\Gamma_j\,.$$

a) By Corollary 2.3.3 a), $K_i(\mathscr{C}_0(\Omega_j\setminus\Gamma_j,F))\approx K_{i+1}(\mathscr{C}_0(\Gamma_j,F))$ for every $j\in J$ so by Corollary 2.3.4 a),

$$K_i(\mathscr{C}(\Omega,F))\approx K_i(F)\times\prod_{j\in J}K_{i+1}(\mathscr{C}_0(\Gamma_j,F))\,.$$

b) By Corollary 2.3.3 b), for every $j\in J$,

$$\Omega_j\setminus\Gamma_j\in\Upsilon\,,\qquad p(\Omega_j\setminus\Gamma_j)=0\,,\qquad q(\Omega_j\setminus\Gamma_j)=\mathit{Card}\,\Gamma_j\,.$$

Thus by Corollary 2.3.4 b),

$$\Omega\in\Upsilon\,,\qquad p(\Omega)=1\,,\qquad q(\Omega)=\sum_{j\in J}\mathit{Card}\,\Gamma_j\,.$$ ■

PROPOSITION 2.3.6 *Let Ω be a compact space belonging to Υ_1, Γ a closed set of Ω, $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$. We use the notation of* the Topological triple (Proposition 2.1.11) *and put there*

$$\Omega_1 := \Omega\,, \qquad \Omega_2 := \Omega \setminus \{\omega_0\}\,, \qquad \Omega_3 := \Omega \setminus \Gamma\,.$$

a) $\Omega \setminus \{\omega_0\}$ *is Υ-null.*

b) $K_i\left(\mathscr{C}_0\left(\Omega \setminus \Gamma, F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\Gamma', F\right)\right)$.

c)

$$0 \longrightarrow K_i(\mathscr{C}\,(\Omega,F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}\,(\Gamma,F)) \underset{\xleftarrow{\Phi_i}}{\xrightarrow{\delta_{1,3,i}}}$$

$$\underset{\xleftarrow{\Phi_i}}{\xrightarrow{\delta_{1,3,i}}} K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma, F)) \longrightarrow 0$$

is a split exact sequence, and the maps

$$K_i(\mathscr{C}\,(\Omega,F)) \times K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma,F)) \longrightarrow K_i(\mathscr{C}\,(\Gamma,F)),$$

$$(a,b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b\,,$$

$$\delta_{2,3,i} : K_i\left(\mathscr{C}_0\left(\Gamma',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma,F))$$

are group isomorphisms.

d) If $\Omega \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of Corollary 2.1.9

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Gamma',F\right)\right) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\Omega \setminus \{\omega_0\},F\right)\right)$$

is a group isomorphism and

$$\Omega \setminus \Gamma, \Gamma' \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = q(\Gamma')\,, \qquad q(\Omega \setminus \Gamma) = p(\Gamma')\,,$$

$$\Phi_{i,(\Omega \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F}\,.$$

e) Assume Γ finite.

e_1) $(\delta_{2,3,i})^{-1} : K_{i+1}(\mathscr{C}_0\,(\Omega \setminus \Gamma,F)) \longrightarrow K_i(F)^{\Gamma'}$ *is a group isomorphism.*

e_2) $\Omega \setminus \Gamma \in \Upsilon\,, \qquad p(\Omega \setminus \Gamma) = 0\,, \qquad q(\Omega \setminus \Gamma) = Card\,\Gamma'\,.$

a) follows from Alexandroff's K-theorem (Theorem 2.2.1 c)).

b) follows from Corollary 2.3.3 a).

c) By a), $\Omega\setminus\{\omega\}$ is K-null and the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) follows from Corollary 2.1.9.

$e_1)$ follows from c) and the Product Theorem (Proposition 2.3.1 a_4)).

$e_2)$ follows from a) and Corollary 2.1.9 c). ■

PROPOSITION 2.3.7 *Let Ω be a locally compact space, Γ a closed set of Ω, $\varphi : \mathscr{C}_0(\Omega\setminus\Gamma,F) \longrightarrow \mathscr{C}_0(\Omega,F)$ the inclusion map,*

$$\psi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}_0(\Gamma,F)\,, \quad x \longmapsto x|\Gamma,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega\setminus\Gamma,F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega,F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma,F) \longrightarrow 0\,.$$

Let $(\Omega_j)_{j\in J}$ be a finite family of pairwise disjoint open sets of Ω the union of which is $\Omega\setminus\Gamma$ and for every $j\in J$ put

$$\psi_j : \mathscr{C}_0\left(\bar{\Omega}_j,F\right) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j\setminus\Omega_j,F\right)\,, \quad x \longmapsto x|(\bar{\Omega}_j\setminus\Omega_j)\,,$$

$$\psi'_j : \mathscr{C}_0(\Omega\setminus\Gamma,F) \longrightarrow \mathscr{C}_0(\Omega_j,F)\,, \quad x \longmapsto x|\Omega_j\,,$$

$$\psi''_j : \mathscr{C}_0(\Gamma,F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j\setminus\Omega_j,F\right)\,, \quad x \longmapsto x|(\bar{\Omega}_j\setminus\Omega_j)$$

and denote by

$$\varphi_j : \mathscr{C}_0(\Omega_j,F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}_j,F\right)\,,$$

$$\varphi'_j : \mathscr{C}_0(\Omega_j,F) \longrightarrow \mathscr{C}_0(\Omega\setminus\Gamma,F)\,,$$

$$\varphi''_j : \mathscr{C}_0(\Omega_j,F) \longrightarrow \mathscr{C}_0(\Omega,F)$$

the inclusion maps and by $\delta_{j,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega_j,F) \xrightarrow{\varphi_j} \mathscr{C}_0\left(\bar{\Omega}_j,F\right) \xrightarrow{\psi_j} \mathscr{C}_0\left(\bar{\Omega}_j\setminus\Omega_j,F\right) \longrightarrow 0\,.$$

a) For every $j \in J$,

$$\delta_{j,i} \circ K_i(\psi''_j) = K_{i+1}(\psi'_j) \circ \delta_i$$

and

$$\delta_i = \sum_{j \in J} K_{i+1}(\varphi'_j) \circ \delta_{j,i} \circ K_i(\psi''_j)\ .$$

b) $K_i(\varphi) = \sum_{j \in J} K_i(\varphi''_j) \circ K_i(\psi'_j)$.

c) Let $j_0 \in J$ such that $\mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right)$ is K-null.

c_1) $K_i(\varphi''_{j_0})$ is a group isomorphism.

c_2) Assume ψ K-null. If we put

$$\Phi_i := K_i(\varphi'_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F))$$

then

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \underset{\Phi_i}{\overset{K_i(\varphi)}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{K_i(\varphi)}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(\mathscr{C}_0(\Gamma, F)) \times K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)),$$

$$(a, b) \longmapsto \delta_{i+1} a + \Phi_i b$$

is a group isomorphism.

a) By the commutativity of the index maps (Axiom 1.2.8),

$$\delta_{j,i} \circ K_i(\psi''_j) = K_{i+1}(\psi'_j) \circ \delta_i\ .$$

Since $\sum_{j \in J} \varphi'_j \circ \psi'_j$ is the identity map of $\mathscr{C}_0(\Omega \setminus \Gamma, F)$,

$$\sum_{j \in J} K_{i+1}(\varphi'_j) \circ \delta_{j,i} \circ K_i(\psi''_j) = \sum_{j \in J} K_{i+1}(\varphi'_j) \circ K_{i+1}(\psi'_j) \circ \delta_i =$$

$$= K_{i+1}\left(\sum_{j \in J} \varphi'_j \circ \psi'_j\right) \circ \delta_i = \delta_i\ .$$

b) We have $\varphi''_j = \varphi \circ \varphi'_j$ for every $j \in J$. Since $\sum_{j\in J} \varphi'_j \circ \psi'_j$ is the identity map of $\mathscr{C}_0(\Omega \setminus \Gamma, F)$,

$$K_i(\varphi) = K_i(\varphi) \circ K_i\left(\sum_{j\in J} \varphi'_j \circ \psi'_j\right) =$$

$$= \sum_{j\in J} K_i(\varphi) \circ K_i(\varphi'_j) \circ K_i(\psi'_j) = \sum_{j\in J} K_i(\varphi''_j) \circ K_i(\psi'_j) .$$

c_1) If we put

$$\bar{\psi} : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right), \quad x \longmapsto x|(\Omega \setminus \Omega_{j_0})$$

then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega_{j_0}, F\right) \xrightarrow{\varphi''_{j_0}} \mathscr{C}_0(\Omega, F) \xrightarrow{\bar{\psi}} \mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. Since $\mathscr{C}_0\left(\Omega \setminus \Omega_{j_0}, F\right)$ is K-null, it follows that $K_i(\varphi''_{j_0})$ is a group isomorphism by the Topological six-term sequence (Proposition 2.1.8 c_1)).

c_2) Since $\varphi \circ \varphi'_{j_0} = \varphi''_{j_0}$,

$$K_i(\varphi) \circ \Phi_i = K_i(\varphi) \circ K_i(\varphi'_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} = K_i(\varphi''_{j_0}) \circ K_i(\varphi''_{j_0})^{-1} = id_{K_i(\mathscr{C}_0(\Omega,F))} .$$

Since ψ is K-null,

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma, F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \underset{\Phi_i}{\overset{K_i(\varphi)}{\rightleftarrows}} K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow 0$$

is a split exact sequence and this implies the last assertion. ■

PROPOSITION 2.3.8 *If $(\Omega_j)_{j\in J}$, $J \neq \emptyset$, is a finite family of compact spaces belonging to Υ_1 then $\prod_{j\in J} \Omega_j \in \Upsilon_1$.*

The assertion follows immediately from Proposition 1.6.5. ■

2.4 Homotopy

PROPOSITION 2.4.1 *Let Ω be a locally compact space, Ω^* its Alexandroff compactification, $(\vartheta_s)_{s\in]0,1]}$ a family of proper continuous maps $\Omega \to \Omega$, and for every $s \in]0,1]$ let $\vartheta^*_s : \Omega^* \longrightarrow \Omega^*$ be the continuous extension of ϑ_s. We assume:*

1) $\Omega^* \times]0,1] \longrightarrow \Omega^*, \quad (\omega, s) \longmapsto \vartheta_s^*(\omega)$ *is continuous,*

2) $\vartheta_1(\omega) = \omega$ *for every* $\omega \in \Omega$*,*

3) for every compact set Γ *of* Ω *there is an* $\varepsilon \in]0,1]$ *with* $\Gamma \cap \vartheta_s(\Omega) = \emptyset$ *for all* $s \in]0, \varepsilon[$.

Then Ω *is null-homotopic and* $\Omega^* \in \Upsilon_1$.

We put for every $s \in [0,1]$,

$$\phi_s : \mathscr{C}_0(\Omega, \mathbb{C}) \longrightarrow \mathscr{C}_0(\Omega, \mathbb{C}), \quad x \longmapsto \begin{cases} x \circ \vartheta_s & \text{if} \quad s \in]0,1] \\ 0 & \text{if} \quad s = 0 \end{cases} .$$

Then $(\phi_s)_{s\in[0,1]}$ is a pointwise continuous path in $\mathscr{C}_0(\Omega, \mathbb{C})$ with $\phi_0 = 0$ and ϕ_1 the identity map of $\mathscr{C}_0(\Omega, F)$. Thus Ω is null-homotopic. By Proposition 1.5.4 d), Ω is Υ-null and by Alexandroff's K-theorem, (Theorem 2.2.1 c)), $\Omega^* \in \Upsilon_1$. ■

COROLLARY 2.4.2 *Let* J *be a set and* $\Omega := [0,1]^J$. *Then* $\Omega \setminus \{0\}$ *is null-homotopic and* $\Omega \in \Upsilon_1$.

The assertion follows from Proposition 2.4.1 by using the map

$$\vartheta : \Omega \times [0,1] \longrightarrow \Omega, \quad (\omega, s) \longmapsto s\omega .$$ ■

PROPOSITION 2.4.3 *Let* Ω *be a locally compact space,* Γ_0, Γ_1 *compact subspaces of* Ω*,* $\vartheta_0 : \Gamma_0 \longrightarrow \Gamma_1$ *a homeomorphism, and* $\vartheta : \Gamma_0 \times [0,1] \longrightarrow \Omega$ *a continuous map such that* $\vartheta(\omega, 0) = \omega$ *and* $\vartheta(\omega, 1) = \vartheta_0(\omega)$ *for every* $\omega \in \Gamma_0$. *We put*

$$\psi_j : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma_j, F), \quad x \longmapsto x|\Gamma_j$$

for every $j \in \{0,1\}$ *and*

$$\varphi : \mathscr{C}(\Gamma_1, F) \longrightarrow \mathscr{C}(\Gamma_0, F), \quad x \longmapsto x \circ \vartheta_0 .$$

a) $K_i(\varphi)$ *is a group isomorphism and* $K_i(\psi_0) = K_i(\varphi) \circ K_i(\psi_1)$.

b) For every $j \in \{0,1\}$ *let* $\varphi_j : \mathscr{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ *be the inclusion map and* $\mathscr{C}(\Gamma_j, F) \overset{\lambda_j}{\to} \mathscr{C}_0(\Omega, F)$ *be a morphism in* $\mathfrak{M}_E$ *such that* $\psi_j \circ \lambda_j = id_{\mathscr{C}(\Gamma_j,F)}$ *and* $\lambda_1 = \lambda_0 \circ \varphi$.

b₁) For every $j \in \{0,1\}$,

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) \xrightarrow{K_i(\varphi_j)} K_i(\mathscr{C}_0(\Omega, F)) \underset{\xleftarrow{K_i(\lambda_j)}}{\xrightarrow{K_i(\psi_j)}}$$

$$\underset{\xleftarrow{K_i(\lambda_j)}}{\xrightarrow{K_i(\psi_j)}} K_i(\mathscr{C}_0(\Gamma_j, F)) \longrightarrow 0$$

is a split exact sequence.

b₂) $Im K_i(\varphi_0) = Im K_i(\varphi_1)$.

b₃) If we put for every $j \in \{0,1\}$

$$\Psi_{j,i} : K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) \longrightarrow Im K_i(\varphi_j)\,, \quad a \longmapsto K_i(\varphi_j)a$$

then $\Psi_{j,i}$ *and*

$$(\Psi_{1,i})^{-1} \circ \Psi_{0,i} : K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F))$$

are well-defined group isomorphisms.

b₄) If $\Omega \setminus \Gamma_0 \in \Upsilon$ *or* $\Omega \setminus \Gamma_1 \in \Upsilon$ *then*

$$\Omega \setminus \Gamma_0, \Omega \setminus \Gamma_1 \in \Upsilon, \qquad p(\Omega \setminus \Gamma_0) = p(\Omega \setminus \Gamma_1)\,, \qquad q(\Omega \setminus \Gamma_0) = q(\Omega \setminus \Gamma_1)\,,$$

$$\Phi_{i,(\Omega\setminus\Gamma_1),F} = (\Psi^F_{1,i})^{-1} \circ \Psi^F_{0,i} \circ \Phi_{i,(\Omega\setminus\Gamma),F}\,.$$

c) If Ω *is compact and if for every* $j \in \{0,1\}$ *there is a continuous map* $\vartheta'_j : \Omega \longrightarrow \Gamma_j$ *such that* $\vartheta'_j(\omega) = \omega$ *for every* $\omega \in \Gamma_j$ *and* $\vartheta_0 \circ \vartheta'_0 = \vartheta'_1$ *then the hypotheses of b) are fulfilled.*

a) For every $s \in [0,1]$ put

$$\nu_s : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\Gamma_0, F)\,, \quad x \longmapsto x(\vartheta(\cdot, s))\,.$$

Then $K_i(\nu_0) = K_i(\nu_1)$ by the homotopy axiom (Axiom 1.2.5). $K_i(\varphi)$ is obviously a group isomorphism. For every $x \in \mathscr{C}_0(\Omega, F)$ and $\omega \in \Gamma_0$,

$$(\nu_0 x)(\omega) = x(\vartheta(\omega, 0)) = x(\omega) = (\psi_0 x)(\omega)\,,$$

$$(\nu_1 x)(\omega) = x(\vartheta(\omega,1)) = x(\vartheta_0(\omega)) = (\psi_1 x)(\vartheta_0(\omega)) = (\varphi\psi_1 x)(\omega),$$

so $\nu_0 = \psi_0$, $\nu_1 = \varphi \circ \psi_1$,

$$K_i(\psi_0) = K_i(\nu_0) = K_i(\nu_1) = K_i(\varphi) \circ K_i(\psi_1) .$$

b_1 follows from the split exact axiom (Axiom 1.2.3).

b_2) Let $j \in \{0,1\}$. We want to prove

$$Im K_i(\varphi_j) = \left\{ c - K_i(\lambda_j) K_i(\psi_j) c \;\middle|\; c \in K_i(\mathscr{C}(\Omega,F)) \right\} .$$

Let $a \in K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F))$ and put $c := K_i(\varphi_j)a$. Then

$$c - K_i(\lambda_j) K_i(\psi_j) c = K_i(\varphi_j) a - K_i(\lambda_j) K_i(\psi_j) K_i(\varphi_j) a = K_i(\varphi_j) a,$$

which proves the "$\subset$ "-inclusion. Let $c \in K_i(\mathscr{C}(\Omega,F))$. Then

$$K_i(\psi_j)(c - K_i(\lambda_j) K_i(\psi_j) c) =$$

$$= K_i(\psi_j) c - K_i(\psi_j) K_i(\lambda_j) K_i(\psi_j) c = K_i(\psi_j) c - K_i(\psi_j) c = 0,$$

$$c - K_i(\lambda_j) K_i(\psi_j) c \in Ker K_i(\psi_j) = Im K_i(\varphi_j) ,$$

which proves the "$\supset$"-inclusion (by b_1)).

Since $\lambda_1 \circ \psi_1 = \lambda_0 \circ \varphi \circ \psi_1$, we get by a),

$$K_i(\lambda_1) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\varphi) \circ K_i(\psi_1) = K_i(\lambda_0) \circ K_i(\psi_0) .$$

Thus, by the above, $Im K_i(\varphi_0) = Im K_i(\varphi_1)$.

b_3) By b_1), $K_i(\varphi_0)$ and $K_i(\varphi_1)$ are injective, the assertion follows from b_2).

b_4) Let $F \xrightarrow{\phi} F'$ be o morphism in $\mathfrak{M}_E$ and for every $j \in \{0,1\}$ put

$$\mu_j : \mathscr{C}_0(\Omega \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Gamma_j, F'\right), \quad x \longmapsto \phi \circ x,$$

$$\mu : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega, F'\right), \quad x \longmapsto \phi \circ x .$$

We mark by a prime the notation associated to F when applied to F'. For every $j \in \{0,1\}$ the diagram

$$\begin{array}{ccc} \mathscr{C}_0(\Omega \setminus \Gamma_j, F) & \xrightarrow{\mu_j} & \mathscr{C}_0(\Omega \setminus \Gamma_j, F') \\ \varphi_j \Big\downarrow & & \Big\downarrow \varphi'_j \\ \mathscr{C}_0(\Omega, F) & \xrightarrow[\mu]{} & \mathscr{C}_0(\Omega, F') \end{array}$$

is commutative. Thus the diagrams

$$\begin{array}{ccc} K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F')) \\ K_i(\varphi_j) \Big\downarrow & & \Big\downarrow K_i(\varphi'_j) \\ K_i(\mathscr{C}_0(\Omega, F)) & \xrightarrow[K_i(\mu)]{} & K_i(\mathscr{C}_0(\Omega, F')) \end{array}$$

$$\begin{array}{ccc} K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F)) & \xrightarrow{K_i(\mu_j)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_j, F')) \\ \Psi_{j,i} \Big\downarrow & & \Big\downarrow \Psi'_{j,i} \\ Im K_i(\varphi_j) & \xrightarrow[\Lambda_i]{} & Im K_i\left(\varphi'_j\right) \end{array}$$

are also commutative, where Λ_i is the map defined by $K_i(\mu)$.

Assume $\Omega \setminus \Gamma_0 \in \Upsilon$ and consider the diagram (by b_2))

$$\begin{array}{ccc} K_i(F)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}(F)^{q(\Omega\setminus\Gamma_0)} & \xrightarrow{\Delta} & A \\ \Phi_{i,(\Omega\setminus\Gamma_0),F} \Big\downarrow & & \Big\downarrow \Phi_{i,(\Omega\setminus\Gamma_0),F'} \\ K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F)) & \xrightarrow{K_i(\mu_0)} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_0, F')) \\ \Psi_{0,i} \Big\downarrow & & \Big\downarrow \Psi'_{0,i} \\ Im K_i(\varphi_0) & \xrightarrow[\Lambda_i]{} & Im K_i(\varphi'_0) \\ \Psi_{1,i} \Big\uparrow & & \Big\uparrow \Psi'_{1,i} \\ K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F)) & \xrightarrow[K_i(\mu_1)]{} & K_i(\mathscr{C}_0(\Omega \setminus \Gamma_1, F')) \end{array}$$

where

$$\Delta := K_i(\phi)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}(\phi)^{q(\Omega\setminus\Gamma_0)} ,$$

$$A := K_i\left(F'\right)^{p(\Omega\setminus\Gamma_0)} \times K_{i+1}\left(F'\right)^{q(\Omega\setminus\Gamma_0)} .$$

By the above, this diagram is commutative and the assertion follows from b_3).

c) For every $j \in \{0,1\}$ put

$$\lambda_j : \mathscr{C}(\Gamma_j, F) \longrightarrow \mathscr{C}(\Omega, F), \quad x \longmapsto x \circ \vartheta'_j .$$

Then $\psi_j \circ \lambda_j = id_{\mathscr{C}(\Gamma_j,F)}$ and for every $x \in \mathscr{C}(\Gamma_1, F)$,

$$\lambda_1 x = x \circ \vartheta'_1 = x \circ \vartheta_0 \circ \vartheta'_0 = (\varphi x) \circ \vartheta'_0 = \lambda_0(\varphi x), \qquad \lambda_1 = \lambda_0 \circ \varphi . \qquad \blacksquare$$

COROLLARY 2.4.4 *Let Ω be a compact space and $\omega, \omega' \in \Omega$ such that there is a continuous path in Ω from ω to ω'.*

a) $K_i(\mathscr{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega'\}, F))$.

b) If $\Omega \setminus \{\omega\} \in \Upsilon$ then

$$\Omega \setminus \{\omega'\} \in \Upsilon, \qquad p(\Omega \setminus \{\omega'\}) = p(\Omega \setminus \{\omega\}), \qquad q(\Omega \setminus \{\omega'\}) = q(\Omega \setminus \{\omega\}) .$$

a) follows from Proposition 2.4.3 b_3) and c).

b) follows from Proposition 2.4.3 b_4) and c). $\blacksquare$

COROLLARY 2.4.5 *Let Ω, Ω' be compact spaces such that $\Omega' \setminus \{\omega'\}$ is null-homotopic for all $\omega' \in \Omega'$, $\omega \in \Omega$, and $\omega'' \in \Omega \times \Omega'$. Then*

$$K_i(\mathscr{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \times \Omega' \setminus \{\omega''\}, F\right)\right) .$$

Let $\omega'' =: (\omega_0, \omega'_0) \in \Omega \times \Omega'$. By Corollary 2.1.10 a),

$$K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega_0\}) \times \Omega', F\right)\right) \approx K_i\left(\mathscr{C}_0\left(\Omega \times \Omega' \setminus \{\omega''\}, F\right)\right)$$

and by Proposition 2.4.3 c),

$$K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega\}) \times \Omega', F\right)\right) \approx K_i\left(\mathscr{C}_0\left((\Omega \setminus \{\omega_0\}) \times \Omega', F\right)\right) .$$

By Proposition 1.4.2 b_3),c),

$$\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times(\Omega'\setminus\{\omega_0'\}),F\right)\approx\mathscr{C}_0\left(\Omega'\setminus\{\omega_0'\},\mathbb{C}\right)\otimes\mathscr{C}_0\left(\Omega\setminus\{\omega\},F\right)$$

is null-homotopic. Since the sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times(\Omega'\setminus\{\omega_0'\}),F\right)\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)$$

$$\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)\longrightarrow\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\{\omega_0'\},F\right)\longrightarrow 0$$

is exact it follows from the topological six-term sequence (Proposition 2.1.8 a_1)),

$$K_i\left(\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\Omega',F\right)\right)\approx$$

$$\approx K_i\left(\mathscr{C}_0\left((\Omega\setminus\{\omega\})\times\{\omega_0'\},F\right)\right)\approx K_i\left(\mathscr{C}_0\left(\Omega\setminus\{\omega\},F\right)\right)\ .$$

■

COROLLARY 2.4.6 *Let Ω be a locally compact space and $\omega_1,\omega_2\in\Omega$ and for every $j\in\{1,2\}$ put*

$$\psi_j:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega_j)\ .$$

If there is a continuous path in Ω from ω_1 to ω_2 then $K_i\left(\psi_1\right)=K_i\left(\psi_2\right)$.

The assertion follows from Proposition 2.4.3 a). ■

COROLLARY 2.4.7 *Let Ω be a locally compact space, Γ a finite subset of Ω, $\omega_0\in\Omega$, and*

$$\psi:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow\mathscr{C}\left(\Gamma,F\right)\,,\quad x\longmapsto x|\Gamma\,,$$

$$\psi_{\omega_0}:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega_0)\ .$$

If for every $\omega\in\Gamma$ there is a continuous path in Ω connecting ω_0 with ω then

$$K_i(\psi):K_i(\mathscr{C}_0\left(\Omega,F\right))\longrightarrow K_i(\mathscr{C}\left(\Gamma,F\right))\approx K_i(F)^{Card\,\Gamma}\,,$$

$$a\longmapsto\left(K_i(\psi_{\omega_0})a\right)_{\omega\in\Gamma}\ .$$

We put

$$\psi_\omega:\mathscr{C}_0\left(\Omega,F\right)\longrightarrow F\,,\quad x\longmapsto x(\omega)$$

for every $\omega\in\Gamma$. By Corollary 2.4.6 , $K_i(\psi_\omega)=K_i(\psi_{\omega_0})$ for every $\omega\in\Gamma$ and the assertion follows from the Product Theorem (Proposition 2.3.1 a)). ■

PROPOSITION 2.4.8 *Let Ω be a path connected compact space, Γ a finite subset of Ω, $\omega_0 \in \Gamma$, $\Gamma' := \Gamma \setminus \{\omega_0\}$,*

$$\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}(\Omega, F),$$

$$\varphi' : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F),$$

$$\varphi'' : \mathscr{C}\left(\Gamma', F\right) \longrightarrow \mathscr{C}(\Gamma, F)$$

the inclusion maps,

$$\psi : \mathscr{C}(\Omega, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

$$\psi' : \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F) \longrightarrow \mathscr{C}\left(\Gamma', F\right), \quad x \longmapsto x|\Gamma',$$

$$\psi_\omega : \mathscr{C}(\Omega, F) \longrightarrow F, \quad x \longmapsto x(\omega)$$

for every $\omega \in \Gamma$, and δ_i, δ_i' the index maps associated to the exact sequences in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}(\Omega, F) \xrightarrow{\psi} \mathscr{C}(\Gamma, F) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi'} \mathscr{C}_0(\Omega \setminus \{\omega_0\}, F) \xrightarrow{\psi'} \mathscr{C}\left(\Gamma', F\right) \longrightarrow 0\,.$$

a) $K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F))$.

b) ψ' is K-null.

c) If we use the group isomorphism of a) then

$$K_i(\psi) : K_i(\mathscr{C}(\Omega, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^\Gamma, \quad (a, b) \longmapsto (a)_{\omega \in \Gamma}\,.$$

d) If we identify $K_i(\mathscr{C}(\Gamma, F))$ with $K_i(F)^\Gamma$ and $K_i(\mathscr{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ then

$$\delta_i : K_i(\mathscr{C}(\Gamma, \cdot)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, \cdot)), \quad (a_\omega)_{\omega \in \Gamma} \longmapsto (\delta_i'(a_\omega - a_{\omega_0}))_{\omega \in \Gamma'}\,.$$

e) Assume $\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)$ K-null.

e_1) $K_i(\psi_{\omega_0}) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(F)$ is a group isomorphism.

e_2) $\delta_i' : K_i(\mathscr{C}(\Gamma', F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ is a group isomorphism.

e_3) If we identify $K_i(\mathscr{C}(\Gamma', F))$ with $K_i(F)^{\Gamma'}$ and $K_i(\mathscr{C}(\Gamma, F))$ with $K_i(F)^\Gamma$ then for all $(a_\omega)_{\omega \in \Gamma'}$

$$K_i(\varphi'')(a_\omega)_{\omega \in \Gamma'} = (a_\omega)_{\omega \in \Gamma},$$

where $a_{\omega_0} = 0$.

e_4) If we identify $K_{i+1}(\mathscr{C}_0(\Omega\setminus\Gamma,F))$ with $K_i(\mathscr{C}(\Gamma',F))$ using $(\delta_i')^{-1}$ of e_2) then for all $(a_\omega)_{\omega\in\Gamma}\in K_i(\mathscr{C}(\Gamma,F))$,

$$\delta_i(a_\omega)_{\omega\in\Gamma}=(a_\omega-a_{\omega_0})_{\omega\in\Gamma'}\,.$$

a) follows from the Alexandroff K-theorem (Theorem 2.2.1 a)).

b) Let $\omega\in\Gamma'$ and let $\vartheta:[0,1]\longrightarrow\Omega$ be a continuous path in Ω connecting ω with ω_0. Then for every $x\in\mathscr{C}_0(\Omega\setminus\{\omega_0\},F)$ the map

$$[0,1]\longrightarrow\mathscr{C}_0(\Omega\setminus\{\omega_0\},F)\,,\quad x\longmapsto x(\vartheta_s(\omega))$$

is continuous. By the homotopy axiom (Axiom 1.2.5) , $K_i(\psi_\omega)=0$ so by the Product Theorem (Proposition 2.3.1 a)), $K_i(\psi')=0$.

c) follows from a), b), and Corollary 2.4.7.

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta_i'=\delta_i\circ K_i(\varphi'')$ so by the Product Theorem (Proposition 2.3.1 a)),

$$\delta_i(0,(a_\omega)_{\omega\in\Gamma'})=\delta_i'(a_\omega)_{\omega\in\Gamma'}$$

for all $(a_\omega)_{\omega\in\Gamma'}\in K_i(F)^{\Gamma'}$. For $a\in K_i(F)$, by c) and by the above,

$$0=\delta_iK_i(\psi)a=\delta_i(a)_{\omega\in\Gamma}=\delta_i(a,(a)_{\omega\in\Gamma'})=$$

$$=\delta_i(a,0)+\delta_i(0,(a)_{\omega\in\Gamma'})=\delta_i(a,0)+\delta_i'(a)_{\omega\in\Gamma'}\,,$$

$\delta_i(a,0)=-\delta_i'(a)_{\omega\in\Gamma'}$. It follows for all $(a_\omega)_{\omega\in\Gamma}$,

$$\delta_i(a_\omega)_{\omega\in\Gamma}=\delta_i(a_{\omega_0},0)+\delta_i(0,(a_\omega)_{\omega\in\Gamma'})=$$

$$=-\delta_i'(a_{\omega_0})_{\omega\in\Gamma'}+\delta_i'(a_\omega)_{\omega\in\Gamma'}=\delta_i'(a_\omega-a_{\omega_0})_{\omega\in\Gamma}\,.$$

e_1) and e_2) follow from the Topological six-term sequence (Proposition 2.1.8) a_1) and b_1), respectively.

e_3) follows from the Product Theorem (Proposition 2.3.1 a)).

e_4) follows from d). ■

EXAMPLE 2.4.9 *Let $n \in \mathbb{N}$. We use the notation of* Proposition 2.4.8 *and put*

$$\Omega := \left\{ re^{\frac{2\pi i j}{n}} \;\middle|\; r \in [0,1],\, j \in \mathbb{N}_n \right\}, \quad \Gamma := \left\{ e^{\frac{2\pi i j}{n}} \;\middle|\; j \in \mathbb{N}_n \right\}, \quad \omega_0 := 1\,.$$

a) $\Omega \setminus \{\omega_0\}$ *is null-homotopic and so K-null.*

b) $K_i(\psi_{\omega_0}) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(F)$ *is a group isomorphism.*

c) $\delta_i' : K_i(\mathscr{C}(\Gamma', F)) \approx K_i(F)^{\Gamma'} \longrightarrow K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ *is a group isomorphism.*

d) If we identify $K_i(\mathscr{C}(\Gamma', F))$ *with* $K_i(F)^{\Gamma'}$ *and* $K_i(\mathscr{C}(\Gamma, F))$ *with* $K_i(F)^{\Gamma}$ (using e.g. Lemma 2.1.4 c)) *then for all* $(a_\omega)_{\omega \in \Gamma'}$

$$K_i(\varphi'')(a_\omega)_{\omega \in \Gamma'} = (a_\omega)_{\omega \in \Gamma}\,,$$

where $a_{\omega_0} = 0$.

e) If we identify $K_{i+1}(\mathscr{C}_0(\Omega \setminus \Gamma, F))$ *with* $K_i(F)^{\Gamma'}$ *using* $(\delta_i')^{-1}$ *of c) then for all* $(a_\omega)_{\omega \in \Gamma}$,

$$\delta_i(a_\omega)_{\omega \in \Gamma} = (a_\omega - a_{\omega_0})_{\omega \in \Gamma'}\,.$$

f) $\Omega \in \Upsilon, \quad p(\Omega) = 1, \quad q(\Omega) = 0, \quad \Phi_{i,\Omega,F} = K_i\left(\psi_{\omega_0}\right), \quad \Omega_\Upsilon = \mathbb{C}_\Upsilon$.

a) By Proposition 2.4.1, $\Omega \setminus \{\omega_0\}$ is null-homotopic.

b) follows from a) and the Topological six-term sequence (Proposition 2.1.8 a)).

c), d), and e) follow from Proposition 2.4.8 b), c), and d), respectively.

f) follows from a) and Proposition 2.4.1. ■

PROPOSITION 2.4.10 *Let Ω be a locally compact spaces, $\omega \in \Omega$, Ω' a compact space, and*

$$\vartheta : \Omega' \times [0,1] \longrightarrow \Omega$$

a continuous map such that $\vartheta(\omega', 0) = \omega$ for all $\omega' \in \Omega'$. Then the map

$$\mathscr{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x \circ \vartheta(\,\cdot\,, 1)$$

is K-null

For every $s \in [0,1]$ put

$$\psi_s : \mathscr{C}_0(\Omega \setminus \{\omega\}, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x \circ \vartheta(\cdot, s)\,.$$

Then for every $x \in \mathscr{C}_0(\Omega \setminus \{\omega\}, F)$ the map

$$[0,1] \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad s \longmapsto \psi_s x$$

is continuous and $\psi_0 x = 0$, so the assertion follows from the homotopy (Axiom 1.2.5). ∎

PROPOSITION 2.4.11 *Let Ω be a locally compact space, Δ a closed set of Ω, Γ a compact set of Δ, $\omega_0 \in \Gamma$ such that $\mathscr{C}_0(\Delta \setminus \{\omega_0\}, F)$ is K-null, and $\vartheta : \Gamma \times [0,1] \longrightarrow \Omega$ a continuous map such that $\vartheta(\omega, 1) = \omega$ and $\vartheta(\omega, 0) = \omega_0$ for all $\omega \in \Gamma$. Then*

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(\mathscr{C}_0(\Gamma \setminus \{\omega_0\}, F))\,.$$

In particular if Γ is finite

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\,\Gamma - 1}\,.$$

We use the notation of the Topological triple (Proposition 2.1.11) and put

$$\Omega_1 := \Omega \setminus \{\omega_0\}, \qquad \Omega_2 := \Omega \setminus \Gamma, \qquad \Omega_3 := \Omega \setminus \Delta\,.$$

By Proposition 2.4.10, $\psi_{1,2}$ is K-null and the first assertion follows from the Topological triple (Proposition 2.1.11 b_4)). The last assertion follows from the first one and from the Product Theorem (Proposition 2.3.1 a)). ∎

Chapter 3

Some Selected Locally Compact Spaces

Throughout this chapter we endow $\{0,1\}$ with a group structure by identifying it with $\mathbb{Z}_2$, F denotes an E-C*-algebra, $i \in \{0,1\}$, and $n \in \mathbb{N}$.

3.1 Balls

DEFINITION 3.1.1 *We put*

$$\mathbb{B}_n := \{ \alpha \in \mathbb{R}^n \mid \|\alpha\| \leq 1 \} \ .$$

THEOREM 3.1.2 *Let Γ be a closed set of $\mathbb{B}_n$, $\omega_0 \in \Gamma$, and $\Gamma' := \Gamma \setminus \{\omega_0\}$.*

a) $\mathbb{B}_n \setminus \{\omega_0\}$ *is null-homotopic and so Υ-null, $\mathbb{B}_n \in \Upsilon_1$, and every exact sequence in $\mathfrak{M}_E$ belongs to $(\mathbb{B}_n)_\Upsilon$. We use in the sequel the notation of* Proposition 2.3.6 *and put there $\Omega := \mathbb{B}_n$.*

b) $K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma', F))$.

c)

$$0 \longrightarrow K_i(\mathscr{C}(\mathbb{B}_n, F)) \xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}(\Gamma, F)) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow 0$$

is a split exact sequence, and the maps

$$K_i(\mathscr{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)),$$

$$(a, b) \longmapsto K_i(\psi_{1,3})a + \Phi_i b\,,$$

$$\delta_{2,3,i} : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F))$$

are group isomorphisms.

d) If $\mathbb{B}_n \setminus \Gamma \in \Upsilon$ or $\Gamma' \in \Upsilon$ then with the notation of Corollary 2.1.9

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Gamma', F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \{\omega_0\}, F))$$

is a group isomorphism and

$$\mathbb{B}_n \setminus \Gamma, \Gamma' \in \Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = q(\Gamma')\,, \qquad q(\mathbb{B}_n \setminus \Gamma) = p(\Gamma')\,,$$

$$\Phi_{i,(\mathbb{B}_n \setminus \Gamma),F} = \delta_{i+1} \circ \Phi_{(i+1),\Gamma',F} \ .$$

e) Assume Γ finite.

e_1)

$$(\delta_{2,3,i})^{-1} : K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \longrightarrow K_i(F)^{\Gamma'}$$

is a group isomorphism.

e_2)

$$K_i(\psi_{1,3}) : K_i(\mathscr{C}(\mathbb{B}_n, F)) \approx K_i(F) \longrightarrow K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^{\Gamma},$$

$$a \longmapsto (a)_{\omega\in\Gamma},$$

and, if we identify $K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ with $K_i(F)^{\Gamma'}$ using the above group isomorphism $(\delta_{2,3,i})^{-1}$, then

$$\delta_{1,3,i} : K_i(\mathscr{C}(\Gamma, F)) \longrightarrow K_i(F)^{Card\,\Gamma'}, \quad (a_\omega)_{\omega\in\Gamma} \longmapsto (a_\omega - a_{\omega_0})_{\omega\in\Gamma'} .$$

e_3)

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = 0, \qquad q(\mathbb{B}_n \setminus \Gamma) = Card\,\Gamma',$$

$$\Phi_{i,(\mathbb{B}_n\setminus\Gamma),F} = \delta_{2,3,(i+1)} \circ \Phi_{(i+1),\Gamma',F},$$

a) Since $\mathbb{B}_n$ is homeomorphic to $[0,1]^n$, it follows from Corollary 2.4.2 that $\mathscr{C}_0(\Omega \setminus \{\omega_0\}, \mathbb{C})$ is null-homotopic and $\mathbb{B}_n \in \Upsilon_1$. By Proposition 1.5.4 d), $\mathbb{B}_n \setminus \{\omega_0\}$ is Υ-null and by Proposition 1.6.6, every exact sequence in $\mathfrak{M}_E$ belongs to $(\mathbb{B}_n)_\Upsilon$.

b), c), d), e_1), and e_3) follow from a) and Proposition 2.3.6.

e_2) follows from a) and Proposition 2.4.8 e_3), e_4). ■

Remark. By b), $K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F))$ depends only on $K_{i+1}(\mathscr{C}_0(\Gamma', F))$ and not on n or on the embedding of Γ in $\mathbb{B}_n$.

COROLLARY 3.1.3 *Let $(\Gamma_j)_{j\in J}$ be a finite family of pairwise disjoint closed sets of $\mathbb{B}_n$, $J \neq \emptyset$, and for every $j \in J$ let $\omega_j \in \Gamma_j$ such that $\mathscr{C}_0(\Gamma_j \setminus \{\omega_j\}, F)$ is K-null. Then*

$$K_i\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j\in J} \Gamma_j, F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \{\, \omega_j \mid j \in J \,\}, F)) \approx K_{i+1}(F)^{Card\,J-1}$$

Put $\Gamma := \bigcup\limits_{j\in J}(\Gamma_j \setminus \{\omega_j\})$,

$$\psi : \mathscr{C}_0\left(\mathbb{B}_n \setminus \left\{ \omega_j \mid j \in J \right\}, F\right) \longrightarrow \mathscr{C}_0(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and denote by $\varphi : \mathscr{C}_0\left(\mathbb{B}_n \setminus \bigcup\limits_{j\in J}\Gamma_j, F\right) \longrightarrow \mathscr{C}_0\left(\mathbb{B}_n \setminus \left\{ \omega_j \mid j \in J \right\}, F\right)$ the inclusion map. Then

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j\in J}\Gamma_j, F\right) \xrightarrow{\varphi} \mathscr{C}_0\left(\mathbb{B}_n \setminus \left\{ \omega_j \mid j \in J \right\}, F\right) \xrightarrow{\psi} \mathscr{C}_0(\Gamma, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. By the Product Theorem (Proposition 2.3.1 c)), $\mathscr{C}_0(\Gamma, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 b)) and Theorem 3.1.2 e_1),

$$K_i\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus \bigcup_{j\in J}\Gamma_j, F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0\left(\mathbb{B}_n \setminus \left\{ \omega_j \mid j \in J \right\}, F\right)) \approx K_{i+1}(F)^{Card J - 1}. \qquad \blacksquare$$

COROLLARY 3.1.4 *Let $(k_j)_{j\in J}$ be a finite family in $\mathbb{N}$ and for every $j \in J$ let Γ_j be a nonempty finite subset of $\mathbb{B}_{k_j}$. If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathbb{B}_{k_j} \setminus \Gamma_j)_{j\in J}$ then*

$$\Omega \in \Upsilon, \qquad p(\Omega) = 1, \qquad q(\Omega) = \sum_{j\in J}(Card\,\Gamma_j - 1).$$

For every $j \in J$ let $\omega_j \in \Gamma_j$. By Theorem 3.1.2 a), $\mathbb{B}_{k_j} \setminus \{\omega_j\}$ is Υ-null and the assertion follows from Corollary 2.3.5 b). $\blacksquare$

COROLLARY 3.1.5 *If Ω is a path connected compact space, $\omega \in \Omega$, and $\omega' \in \mathbb{B}_n \times \Omega$ then*

$$K_i(\mathscr{C}_0(\Omega \setminus \{\omega\}, F)) \approx K_i\left(\mathscr{C}_0\left(\mathbb{B}_n \times \Omega \setminus \{\omega'\}, F\right)\right).$$

B¡ Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_n \setminus \{\omega_0\}, F)$ is K-null for every $\omega_0 \in \mathbb{B}_n$ and the assertion follows from Corollary 2.4.5. $\blacksquare$

COROLLARY 3.1.6 *Let* Γ *be a closed set of* $\mathbb{B}_n$ *and* Ω *an open set of* $\mathbb{B}_n$, $\Omega \subset \Gamma$. *Then for all* $\omega \in \Gamma \setminus \Omega$,

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) \,,$$

$$K_i(\mathscr{C}_0(\Gamma \setminus \Omega, F)) \approx K_i(\mathscr{C}(\Gamma, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) \,.$$

By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \,,$$

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F))$$

and by the Product Theorem (Proposition 2.3.1a)),

$$K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus (\Gamma \setminus \Omega), F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \Gamma, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) \,,$$

so

$$K_i(\mathscr{C}_0((\Gamma \setminus \Omega) \setminus \{\omega\}, F)) \approx K_i(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}_0(\Omega, F)) \,.$$

The last relation follows from the Alexandroff K-theorem (Proposition 2.2.1 a)). ■

COROLLARY 3.1.7 *If* Ω *is an open set of* $\mathbb{B}_n$, $\Omega \neq \mathbb{B}_n$, *and* Γ *a compact set of* Ω *then*

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) \,.$$

Let $\omega \in \mathbb{B}_n \setminus \Omega$. By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(\mathscr{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) \,,$$

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_{i+1}(\mathscr{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) \,.$$

By the Product Theorem (Proposition 2.3.1 a)),

$$K_{i+1}(\mathscr{C}_0(((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}) \cup \Gamma, F)) \approx$$

$$\approx K_{i+1}(\mathscr{C}_0((\mathbb{B}_n \setminus \Omega) \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) \,,$$

so

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma, F)) \approx K_i(\mathscr{C}_0(\Omega, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)) \,. \quad ■$$

3.2 Euclidean Spaces and Spheres

DEFINITION 3.2.1 *We put*

$$\mathbb{S}_{n-1} := \{ \alpha \in \mathbb{R}^n \mid \|\alpha\| = 1 \} , \qquad \mathbb{T} := \mathbb{S}_1 .$$

THEOREM 3.2.2

a) $\mathbb{R}^n \in \Upsilon , \qquad p(\mathbb{R}^n) = \frac{1+(-1)^n}{2} , \qquad q(\mathbb{R}^n) = \frac{1-(-1)^n}{2} ,$

$$\mathbb{R}_\Upsilon \subset (\mathbb{R}^n)_\Upsilon , \qquad K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F) .$$

b) $\mathbb{S}_n \in \Upsilon , \quad p(\mathbb{S}_n) = \frac{3+(-1)^n}{2} , \quad q(\mathbb{S}_n) = \frac{1-(-1)^n}{2} , \quad \mathbb{R}_\Upsilon \subset (\mathbb{S}_n)_\Upsilon ,$

$$K_i(\mathscr{C}(\mathbb{S}_n, F)) \approx$$

$$\approx \begin{cases} K_i(F)^2 & \text{if } n \text{ is even} \\ K_i(F) \times K_{i+1}(F) & \text{if } n \text{ is odd} \end{cases} = K_i(F) \times K_{i+n}(F) ,$$

and the map

$$K_i(F) \times K_{i+n}(F) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_n, F)), \quad (a,b) \longmapsto K_i(\lambda)a + K_{i+n}(\varphi)b$$

is a group isomorphism, where $\varphi : \mathscr{C}_0(\mathbb{R}^n, F) \approx K_{i+n}(F) \longrightarrow \mathscr{C}(\mathbb{S}_n, F)$ *denotes the inclusion map and*

$$\lambda : F \longrightarrow \mathscr{C}(\mathbb{S}_n, F) , \quad x \longmapsto x1_{\mathscr{C}(\mathbb{S}_n, \mathbb{C})} .$$

c) Let Γ *be a closed set of* $\mathbb{R}^n$, $\Gamma \neq \mathbb{R}^n$.

c_1) The map

$$\mathscr{C}_0(\mathbb{R}^n, F) \longrightarrow \mathscr{C}_0(\Gamma, F) , \quad x \longmapsto x|\Gamma$$

is K-null.

c_2) If Γ *is compact then*

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}(\Gamma, F)) .$$

If in addition $\Gamma \in \Upsilon$ *then* $\mathbb{R}^n \setminus \Gamma \in \Upsilon$, *and*

$$p(\mathbb{R}^n \setminus \Gamma) = \begin{cases} q(\Gamma)+1 & \text{if } n \text{ is even} \\ q(\Gamma) & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbb{R}^n \setminus \Gamma) = \begin{cases} p(\Gamma) & \text{if } n \text{ is even} \\ p(\Gamma)+1 & \text{if } n \text{ is odd} \end{cases}.$$

d) If Γ *is finite then* $\mathbb{R}^n \setminus \Gamma \in \Upsilon$, *and*

$$p(\mathbb{R}^n \setminus \Gamma) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbb{R}^n \setminus \Gamma) = \begin{cases} \operatorname{Card}\Gamma & \text{if } n \text{ is even} \\ \operatorname{Card}\Gamma+1 & \text{if } n \text{ is odd} \end{cases}.$$

e) Let Γ *be a closed set of* $\mathbf{S}_n$, $\Gamma \neq \mathbf{S}_n$, $\omega \in \Gamma$, *and* $\Gamma' := \Gamma \setminus \{\omega\}$.

e_1*)* $K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}_0(\Gamma \setminus \{\omega\}, F))$.

e_2*) If* $\Gamma' \in \Upsilon$ *then* $\mathbf{S}_n \setminus \Gamma \in \Upsilon$, *and*

$$p(\mathbf{S}_n \setminus \Gamma) = \begin{cases} q(\Gamma')+1 & \text{if } n \text{ is even} \\ q(\Gamma') & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbf{S}_n \setminus \Gamma) = \begin{cases} p(\Gamma') & \text{if } n \text{ is even} \\ p(\Gamma')+1 & \text{if } n \text{ is odd} \end{cases}.$$

e_3*) If* Γ *is finite, then* $\mathbf{S}_n \setminus \Gamma \in \Upsilon$, *and*

$$p(\mathbf{S}_n \setminus \Gamma) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

$$q(\mathbf{S}_n \setminus \Gamma) = \begin{cases} \operatorname{Card}\Gamma' & \text{if } n \text{ is even} \\ \operatorname{Card}\Gamma & \text{if } n \text{ is odd} \end{cases}.$$

f) If $m \in \mathbb{N}, m < n$, *then*

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_i(F) \times K_{i+n-m+1}(F).$$

g) For $m \in \mathbb{N}, m < n$,

$$K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_m, F)) \approx K_{i+m+1}(F).$$

a) Since $\mathbb{R}$ is homeomorphic to $]0,1[= \mathbb{B}_1 \setminus \{-1,1\}$ we get

$$\mathbb{R} \in \Upsilon, \qquad p(\mathbb{R}) = 0, \qquad q(\mathbb{R}) = 1$$

from Theorem 3.1.2 e_3) and the assertion follows from Corollary 1.5.12.

b) Since $\boldsymbol{S}_n$ is homeomorphic to the Alexandroff compactification of $\mathbb{R}^n$, b) follows from a) and the Alexandroff K-theorem (Theorem 2.2.1 a),b)).

c1) We may assume $0 \in \mathbb{R}^n \setminus \Gamma$. Put

$$\vartheta : \Gamma \times]0,1] \longrightarrow \mathbb{R}^n, \quad (\omega, s) \longmapsto \frac{1}{s}\,\omega$$

and for every $s \in [0,1]$

$$\psi_s : \mathscr{C}_0(\mathbb{R}^n, F) \longrightarrow \mathscr{C}_0(\Gamma, F), \quad x \longmapsto \begin{cases} x \circ \vartheta(\cdot, s) & \textit{if} \quad s \neq 0 \\ 0 & \textit{if} \quad s = 0 \end{cases}.$$

Then for every $x \in \mathscr{C}_0(\mathbb{R}^n, F)$,

$$[0,1] \longrightarrow \mathscr{C}_0(\Gamma, F), \quad s \longmapsto \psi_s x$$

is continuous, $\psi_1 x = x|\Gamma$, and $\psi_0 x = 0$. Thus the assertion follows from the homotopy axiom (Axiom 1.2.5).

c2) We identify the homeomorphic spaces $\{\, \alpha \in \mathbb{R}^n \mid \|\alpha\| < 1 \,\}$ and $\mathbb{R}^n$, put $\omega := (1,0,\cdots,0) \in \mathbb{B}_n$ and

$$\psi : \mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\}, F) \longrightarrow \mathscr{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F), \quad x \longmapsto x|((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma),$$

and denote by $\varphi : \mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\}, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\}, F) \xrightarrow{\psi} \mathscr{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F) \longrightarrow 0\,.$$

By Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\}, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)), the map

$$\delta_{i+1} : K_{i+1}(\mathscr{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F))$$

is a group isomorphism. By the Product Theorem (Proposition 2.3.1 a),b)),

$$K_{i+1}(\mathscr{C}_0((\mathbf{S}_{n-1} \setminus \{\omega\}) \cup \Gamma, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}(\Gamma, F)),$$

and $\Gamma \in \Upsilon$ implies $\mathbb{R}^n \setminus \Gamma \in \Upsilon$. By a), $K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \{\omega\}, F)) \approx K_{i+n}(F)$ so

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \approx K_{i+n}(F) \times K_{i+1}(\mathscr{C}(\Gamma, F))$$

as well as the last assertions.

d) follows from c) and the Product Theorem (Proposition 2.3.1 a),b)).

e) $\mathbf{S}_n \setminus \Gamma$ is homeomorphic to $\mathbb{R}^n \setminus (\Gamma \setminus \{\omega\})$ and the assertion follows from c) and d).

f) Step 1

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F))$$

Let $\omega \in \mathbf{S}_m$. Then $\mathbf{S}_n \setminus \mathbf{S}_m = (\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$. Since $(\mathbf{S}_n \setminus \{\omega\}) \setminus (\mathbf{S}_m \setminus \{\omega\})$ is homeomorphic to $\mathbb{R}^n \setminus \mathbb{R}^m$ we get

$$K_i(\mathscr{C}_0(\mathbf{S}_n \setminus \mathbf{S}_m, F)) \approx K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) .$$

Step 2

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_i(F) \times K_{i+n-m+1}(F)$$

We identify $\mathbb{R}^n \setminus \mathbb{R}^m$ with $\left\{ \alpha \in \mathbb{B}_n \mid \|\alpha\| < 1\,, \sum_{j=m+1}^{n} \alpha_j^2 \neq 0 \right\}$, put

$$\psi : \mathbb{B}_n \setminus \mathbb{B}_m \longrightarrow \mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}\,, \quad x \longmapsto x|(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1})\,,$$

and denote by $\varphi : \mathbb{R}^n \setminus \mathbb{R}^m \longrightarrow \mathbb{B}_n \setminus \mathbb{B}_m$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F) \xrightarrow{\varphi} \mathscr{C}_0(\mathbb{B}_n \setminus \mathbb{B}_m, F) \xrightarrow{\psi} \mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F) \longrightarrow 0 .$$

By Proposition 2.4.1, $\mathscr{C}_0(\mathbb{B}_n \setminus \mathbb{B}_m, F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)) and Step 1,

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_{m-1}, F)) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^{n-1} \setminus \mathbb{R}^{m-1}, F\right)\right) .$$

For $m = 1$, by e_1),

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbf{S}_{n-1} \setminus \mathbf{S}_0, F)) \approx K_{i+n}(F) \times K_i(F) .$$

By induction and by the above,

$$K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \mathbb{R}^m, F)) \approx K_{i+n-m+1}\left(\mathscr{C}_0\left(\mathbb{R}^{n-m+1} \setminus \mathbb{R}, F\right)\right) \approx$$

$$\approx K_{i+n-m+1}(F) \times K_i(F) \, .$$

g) Let $\omega \in \mathbb{S}_m$. Since $\mathbb{S}_m \setminus \{\omega\}$ is homeomorph to $\mathbb{R}^m$, by a),

$$K_i(\mathscr{C}_0(\mathbb{S}_m \setminus \{\omega\}, F)) \approx K_{i+m}(F) \, .$$

By Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbb{S}_m, F)) \approx K_{i+1}(\mathscr{C}_0(\mathbb{S}_m \setminus \{\omega\}, F)) \approx K_{i+1+m}(F) \, . \qquad \blacksquare$$

EXAMPLE 3.2.3 *Put*

$$\Omega_1 := \mathbb{S}_1 \cup \left\{ re^{\frac{2\pi i j}{n}} \;\middle|\; r \in [0,1],\ j \in \mathbb{N}_n \right\},$$

$$\Omega_2 := \mathbb{S}_2 \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_3 = 0 \} \cup \{ \alpha \in \mathbb{B}_3 \mid \alpha_1 = \alpha_2 = 0 \} \, ,$$

$$\Omega_3 := \mathbb{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{ \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \} \right) .$$

a) $K_i(\mathscr{C}(\Omega_1, F)) = K_i(F) \times K_{i+1}(F)^n$.

b) $K_i(\mathscr{C}(\Omega_2, F)) \approx K_i(F)^3 \times K_{i+1}(F)^2$.

c) $K_i(\mathscr{C}(\Omega_3, F)) = K_i(F) \times K_{i+n+1}(F)^{2^n}$.

a) By Theorem 3.2.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F)^n$$

and by Theorem 3.1.2 a),b),c),

$$K_i(\mathscr{C}(\Omega_1, F)) \approx K_i(\mathscr{C}(\mathbb{B}_2, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega_1, F)) \approx K_i(F) \times K_{i+1}(F)^n \, .$$

b) By Theorem 3.2.2 a),b),

$$\mathbb{R}^2, \mathbb{S}_1 \in \Upsilon, \qquad p(\mathbb{R}^2) = 1\,, \qquad q(\mathbb{R}^2) = 0\,, \qquad p(\mathbb{S}_1) = 1\,, \qquad q(\mathbb{S}_1) = 1\,,$$

so by Corollary 1.5.11 d_1),

$$K_i\left(\mathscr{C}_0\left(\mathbb{R}^2\times\mathbf{S}_1,F\right)\right)\approx K_i(F)\times K_{i+1}(F)\ .$$

Since $\mathbb{B}_3\setminus\Omega_2$ is homeomorphic to the topological sum of two copies of $\mathbb{R}^2\times\mathbf{S}_1$ we get by the Product Theorem (Proposition 2.3.1 a))

$$K_i(\mathscr{C}_0(\mathbb{B}_3\setminus\Omega_2,F))\approx K_i(F)^2\times K_{i+1}(F)^2\ .$$

By Theorem 3.1.2 a),b),c),

$$K_i(\mathscr{C}(\Omega_2,F))\approx K_i(\mathscr{C}(\mathbb{B}_3,F))\times K_{i+1}(\mathscr{C}_0(\mathbb{B}_3\setminus\Omega_2,F))\approx K_i(F)^3\times K_{i+1}(F)^2\,.$$

c) By Theorem 3.2.2 a), $K_i(\mathscr{C}_0(\mathbb{R}^n,F))\approx K_{i+n}(F)$. Since $\mathbb{B}_n\setminus\Omega_3$ is homeomorphic to the topological sum of 2^n copies of $\mathbb{R}^n$, we get by the Product Theorem (Proposition 2.3.1 a)) $K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega_3,F))\approx K_{i+n}(F)^{2^n}$. By Theorem 3.1.2 a),b),c),

$$K_i(\mathscr{C}(\Omega_3,F))\approx K_i(\mathscr{C}(\mathbb{B}_n,F))\times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega_3,F))\approx$$

$$\approx K_i(F)\times K_{i+n+1}(F)^{2^n}\ .$$ ■

Remark. The above a) and b) will be generalized in Example 3.5.11 b) and c), respectively.

COROLLARY 3.2.4 *Let $(k_j)_{j\in J}$ be a finite family in $\mathbb{N}$ and*

$$p:=Card\left\{\,j\in J\mid k_j\text{ is even}\right\},\qquad q:=Card\left\{\,j\in J\mid k_j\text{ is odd}\right\}\,.$$

a) If Ω denotes the Alexandroff compactification of the topological sum of the family $(\mathbb{R}^{k_j})_{j\in J}$ then

$$\Omega\in\Upsilon,\qquad \mathbb{R}_\Upsilon\subset\Omega_\Upsilon,\qquad p(\Omega)=p+1,\qquad q(\Omega)=q\,.$$

b) For every $j\in J$ let $\omega_j\in\mathbf{S}_{k_j}$ and let Ω' denote the compact space obtained from the topological sum of the family $(\mathbf{S}_{k_j})_{j\in J}$ by identifying all the points of the family $(\omega_j)_{j\in J}$. If $J\neq\emptyset$ then

$$\Omega'\in\Upsilon,\qquad \mathbb{R}_\Upsilon\subset\Omega'_\Upsilon,\qquad p(\Omega')=p+1,\qquad q(\Omega')=q\,.$$

In particular if $k_j=1$ for all $j\in J$ then $p(\Omega')=1$, $q(\Omega')=Card\,J$.

a) By Theorem 3.2.2 a), $\mathbb{R}^{k_j} \in \Upsilon$, $\mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon$,

$$p\left(\mathbb{R}^{k_j}\right) = \begin{cases} 1 & \text{if} \quad k_j \text{ is even} \\ 0 & \text{if} \quad k_j \text{ is odd} \end{cases}, \qquad q\left(\mathbb{R}^{k_j}\right) = \begin{cases} 0 & \text{if} \quad k_j \text{ is even} \\ 1 & \text{if} \quad k_j \text{ is odd} \end{cases}$$

for every $j \in J$. The assertion follows now from the Product Theorem (Proposition 2.3.1 b)) and from Alexandroff's K-theorem (Proposition 2.2.1 b)).

b) follows from a) since Ω and Ω' are homeomorphic. ■

COROLLARY 3.2.5 *Let $(k_j)_{j\in J}$ be a finite family in* $\mathbb{N}$,

$$p := Card \left\{ j \in J \mid k_j \text{ is even} \right\}, \qquad q := Card \left\{ j \in J \mid k_j \text{ is odd} \right\},$$

$(\Gamma_j)_{j\in J}$ a pairwise disjoint family of closed sets of $\mathbb{B}_n$ such that Γ_j is homeomorphic to $\mathbf{S}_{k_j}$ for every $j \in J$, and $\Gamma := \bigcup_{j\in J} \Gamma_j$. Then

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = q, \qquad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1 .$$

By Theorem 3.2.2 a),b), for $j \in J$,

$$\mathbb{R}^{k_j}, \mathbf{S}_{k_j} \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{R}^{k_j})_\Upsilon \cap (\mathbf{S}_{k_j})_\Upsilon,$$

$$p\left(\mathbb{R}^{k_j}\right) = \frac{1+(-1)^{k_j}}{2}, \qquad q\left(\mathbb{R}^{k_j}\right) = \frac{1-(-1)^{k_j}}{2},$$

$$p\left(\mathbf{S}_{k_j}\right) = \frac{3+(-1)^{k_j}}{2}, \qquad q\left(\mathbf{S}_{k_j}\right) = \frac{1-(-1)^{k_j}}{2}.$$

Let $\omega \in \Gamma$ and $\Gamma' := \Gamma \setminus \{\omega\}$. By the Product Theorem (Proposition 2.3.1 b)),

$$\Gamma' \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset \Gamma'_\Upsilon, \qquad p(\Gamma') = 2p - 1, \qquad q(\Gamma') = q,$$

so by Theorem 3.1.2 d),

$$\mathbb{B}_n \setminus \Gamma \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset (\mathbb{B}_n \setminus \Gamma)_\Upsilon, \qquad p(\mathbb{B}_n \setminus \Gamma) = q, \qquad q(\mathbb{B}_n \setminus \Gamma) = 2p - 1 .$$ ■

COROLLARY 3.2.6 *If Ω is a connected closed set of $\mathbb{B}_2$ possessing a triangulation with r_0 vertices, r_1 chords, and r_2 triangles then*

$$K_i\left(\mathscr{C}\left(\Omega, F\right)\right) \approx K_i\left(F\right) \times K_{i+1}\left(F\right)^{1-r_0+r_1-r_2}.$$

Sketch of a proof. If Ω has k holes then $r_0 - r_1 + r_2 + k = 1$. By Theorem 3.1.2 c),

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega, F)) \ .$$

By Theorem 3.2.2 a) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \Omega, F)) \approx K_i(F)^k$$

so

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times K_{i+1}(F)^{1-r_0+r_1-r_2} \ . \qquad \blacksquare$$

COROLLARY 3.2.7 *We identify the homeomorphic spaces* $\mathbb{R}^n$ *and*

$$\{ \alpha \in \mathbb{R}^n \mid \|\alpha\| < 1 \} \ .$$

Let Γ *be a finite subset of* $\mathbb{R}^n$, Δ *a subset of* Γ, $\omega \in \Delta$, $\Gamma' := \Gamma \setminus \{\omega\}$, $\Delta' := \Delta \setminus \{\omega\}$. *We use the notation of the* Topological triple (Proposition 2.1.11) *and put*

$$\Omega_1 := \mathbb{B}_n \setminus \{\omega\}, \qquad \Omega_2 := \mathbb{R}^n \setminus \Delta, \qquad \Omega_3 := \mathbb{R}^n \setminus \Gamma .$$

a) $\delta_{1,2,i}$ *and* $\delta_{1,3,i}$ *are group isomorphisms.*

b) $\psi_{2,3}$ *is K-null.*

c) If we put $\Phi_i := \delta_{1,3,(i+1)} \circ K_{i+1}(\varphi') \circ (\delta_{1,2,(i+1)})^{-1}$ *then*

$$0 \longrightarrow K_{i+1}(\mathscr{C}(\Gamma \setminus \Delta, F)) \xrightarrow{\delta_{2,3,(i+1)}} K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)) \underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{K_i(\varphi_{2,3})}{\rightleftarrows}} K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow 0$$

is a split exact sequence and the map

$$K_{i+1}(\mathscr{C}(\Gamma \setminus \Delta, F)) \times K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Delta, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n \setminus \Gamma, F)),$$

$$(a, b) \longmapsto \delta_{2,3,(i+1)} a + \Phi_i b$$

is a group isomorphism.

By Theorem 3.1.2 a), $\mathscr{C}_0(\Omega_1,F)$ is K-null and by Proposition 2.4.10, $\psi_{2,3}$ is K-null. By the Product Theorem (Proposition 2.3.1 a)),

$$K_i(\psi\circ\varphi') = id_{K_i(\mathscr{C}_0(\Omega_1\setminus\Omega_2,F))}$$

and a) and c) follow from the Topological triple (Proposition 2.1.11 c)). ■

COROLLARY 3.2.8 *Let $\omega\in\mathbf{S}_{n-1}$. We use the notation of the* Topological triple (Proposition 2.1.11) *and put*

$$\Omega_1 := \mathbb{B}_n\,,\qquad \Omega_2 := \mathbb{B}_n\setminus\{\omega\}\,,\qquad \Omega_3 := \mathbb{B}_n\setminus\mathbf{S}_{n-1}\,.$$

a) $\varphi_{1,3}$ is K-null.

b) $\delta_{2,3,i} : K_i(\mathscr{C}_0(\mathbf{S}_{n-1}\setminus\{\omega\},F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},F))$ is a group isomorphism.

c) If we put $\Phi_i := K_i(\varphi)\circ(\delta_{2,3,i})^{-1}$ then

$$0\longrightarrow K_i(\mathscr{C}(\mathbb{B}_n,F))\xrightarrow{K_i(\psi_{1,3})} K_i(\mathscr{C}(\mathbf{S}_{n-1},F)) \underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}}$$

$$\underset{\Phi_i}{\overset{\delta_{1,3,i}}{\rightleftarrows}} K_{i+1}(\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},F))\longrightarrow 0$$

is a split exact sequence and the map

$$K_i(\mathscr{C}(\mathbb{B}_n,F))\times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},F))\longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1},F)),$$

$$(a,b)\longmapsto K_i(\psi_{1,3})a+\Phi_i b$$

is a group isomorphism.

d) Let $\phi : G\longrightarrow H$ be a morphism in $\mathfrak{M}_E$ and put

$$\phi_{\mathbb{B}} : \mathscr{C}(\mathbb{B}_n,G)\longrightarrow\mathscr{C}(\mathbb{B}_n,H)\,,\quad x\longmapsto\phi\circ x\,,$$

$$\phi_{\mathbb{S}} : \mathscr{C}(\mathbf{S}_{n-1},G)\longrightarrow\mathscr{C}(\mathbf{S}_{n-1},H)\,,\quad x\longmapsto\phi\circ x\,,$$

$$\phi_{\mathbb{B},\mathbb{S}} : \mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},G)\longrightarrow\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},H)\,,\quad x\longmapsto\phi\circ x\,.$$

If we identify $K_i(\mathscr{C}(\mathbf{S}_{n-1},F))$ with

$$K_i(\mathscr{C}(\mathbb{B}_n,F))\times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n\setminus\mathbf{S}_{n-1},F))$$

for $F \in \{\mathrm{G}, \mathrm{H}\}$ using the isomorphism of c) then

$$K_i(\phi_{\mathbb{S}}) : K_i(\mathscr{C}(\mathbf{S}_{n-1}, G)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1}, H)),$$

$$(a,b) \longmapsto (K_i(\phi_{\mathbb{B}})a, K_{i+1}(\phi_{\mathbb{B},\mathbb{S}})b) .$$

By Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_n \setminus \{\omega\}, F)$ is K-null and the assertion follows from the Topological triple (Proposition 2.1.11 a)) and Corollary 2.1.12 b). ■

PROPOSITION 3.2.9 *Put*

$$\Omega := \mathbb{B}_{n+1} \setminus \{\, \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0 \,\} ,$$

$$\Omega' := \mathbf{S}_n \setminus \{\, \alpha \in \mathbf{S}_n \mid \alpha_{n+1} = 0 \,\} ,$$

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0\left(\Omega', F\right) , \quad x \longmapsto x|\Omega'$$

and denote by

$$\varphi : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathscr{C}_0(\Omega, F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0\left(\Omega', F\right) \longrightarrow 0 .$$

a)

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+n}(F) , \qquad K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \approx K_{i+n}(F)^2 ,$$

$$K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n}(F) .$$

b) If we identify the groups of a) then

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) , \quad (a,b) \longmapsto a + b ,$$

$$0 \longrightarrow K_i(\mathscr{C}_0(\Omega, \cdot)) \xrightarrow{K_i(\psi)} K_i\left(\mathscr{C}_0\left(\Omega', \cdot\right)\right) \xrightarrow{\delta_i} K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, \cdot)) \longrightarrow 0$$

is an exact sequence, and there is a group automorphism $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ such that

$$K_i(\psi) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) , \quad a \longmapsto (\Phi_i a, -\Phi_i a) .$$

c) If

$$\lambda' : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\mathbb{B}_{n+1}, F),$$

$$\lambda'' : \mathscr{C}_0(\Omega', F) \longrightarrow \mathscr{C}(\mathbf{S}_n, F)$$

denote the inclusion maps and if we identify $K_i(\mathscr{C}_0(\Omega', F))$ with $K_{i+n}(F)^2$ using a) and $K_i(\mathscr{C}(\mathbf{S}_n, F))$ with $K_i(F) \times K_{i+n}(F)$ using Theorem 3.2.2 b) *then λ' is K-null and*

$$K_i(\lambda'') : K_i(\mathscr{C}(\Omega', F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_n, F)), \quad (a,b) \longmapsto (0, a+b).$$

a) By Theorem 3.2.2 a), $K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1} \setminus \mathbf{S}_n$ is homeomorphic to $\mathbb{R}^{n+1}$, $K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F)) \approx K_{i+n}(F)$. Since Ω' is homeomorphic to the topological sum of $\mathbb{R}^n$ and $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega', F)) \approx K_{i+n}(F)^2$ by the Product Theorem (Proposition 2.3.1 a)). Put

$$\Gamma := \{ \alpha \in \Omega \mid \alpha_{n+1} = 0 \}$$

and for every $s \in]0,1]$

$$\vartheta_s : \Omega \setminus \Gamma \longrightarrow \Omega \setminus \Gamma, \quad (\alpha_j)_{j \in \mathbb{N}_{n+1}} \longmapsto ((\alpha_j)_{j \in \mathbb{N}_n}, s\alpha_{n+1}).$$

By Proposition 2.4.1, $\mathscr{C}_0(\Omega \setminus \Gamma, F)$ is K-null, so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\mathscr{C}_0(\Omega, F)) \approx K_i(\mathscr{C}_0(\Gamma, F))$. Since Γ is homeomorphic to $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+n}(F)$ by the above.

b) Put $\omega := (1, 0, \cdots, 0) \in \mathbb{B}_{n+1}$,

$$\psi' : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \longrightarrow \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F), \quad x \longmapsto x|(\mathbf{S}_n \setminus \{\omega\}),$$

and denote by

$$\varphi' : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F),$$

$$\varphi'' : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F)$$

$$\varphi''' : \mathscr{C}_0(\Omega', F) \longrightarrow \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F)$$

the inclusion maps and by δ'_i the six-term sequence index maps associated with the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi'} \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \{\omega\}, F) \xrightarrow{\psi'} \mathscr{C}_0(\mathbf{S}_n \setminus \{\omega\}, F) \longrightarrow 0.$$

By Theorem 3.1.2 a), $\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\{\omega\},F)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 c)),

$$\delta_i':K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F))\longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F))$$

is a group isomorphism. By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \delta_i'\circ K_i(\varphi''')$. Thus if we identify the above groups using δ_i' then δ_i is identified with $K_i(\varphi''')$. By Corollary 2.3.2

$$K_i(\varphi'''):K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right)\longrightarrow K_i(\mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F))\,,\quad (a,b)\longmapsto a+b\,.$$

Since $\mathbf{S}_n\setminus\{\omega\}$ is homeomorphic to $\mathbb{R}^n$, we get

$$\delta_i:K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right)\longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F))\,,\quad (a,b)\longmapsto a+b\,.$$

Thus δ_i is surjective and the other assertions follow from the six-term axiom (Axiom 1.2.7).

c) λ' is K-null since it factorizes through null (Theorem 3.1.2 a)). Put $\omega:=(1,0,\cdots,0)\in\mathbb{B}_{n+1}$ and denote by

$$\lambda''':\mathscr{C}_0(\mathbf{S}_n\setminus\{\omega\},F)\longrightarrow\mathscr{C}(\mathbf{S}_n,F)$$

the inclusion map. By the proof of b), since $\lambda''=\lambda'''\circ\varphi'''$,

$$K_i\left(\lambda''\right):K_i\left(\mathscr{C}\left(\Omega',F\right)\right)\longrightarrow K_i(\mathscr{C}(\mathbf{S}_n,F))\,,\quad (a,b)\longmapsto(0,a+b)$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.2.10 *Let Γ be a closed set of $\mathbb{R}^n$, $\Gamma\neq\mathbb{R}^n$,*

$$\varphi:\mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F)\longrightarrow\mathscr{C}_0(\mathbb{R}^n,F)$$

the inclusion map,

$$\psi:\mathscr{C}_0(\mathbb{R}^n,F)\longrightarrow\mathscr{C}_0(\Gamma,F)\,,\quad x\longmapsto x|\Gamma,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F)\xrightarrow{\varphi}\mathscr{C}_0(\mathbb{R}^n,F)\xrightarrow{\psi}\mathscr{C}_0(\Gamma,F)\longrightarrow 0\,.$$

a) ψ *is K-null.*

b) The sequence

$$0 \longrightarrow K_{i+1}(\mathscr{C}_0(\Gamma,F)) \xrightarrow{\delta_{i+1}} K_i(\mathscr{C}_0(\mathbb{R}^n\setminus\Gamma,F)) \xrightarrow{K_i(\varphi)} \mathscr{C}_0(\Gamma,F) \longrightarrow 0$$

is exact.

c) Let $(\Omega_j)_{j\in J}$ *be a finite family of pairwise disjoint open sets of* $\mathbb{R}^n$ *the union of which is* $\mathbb{R}^n\setminus\Gamma$. *If there is a* $j_0\in J$ *such that* $\mathscr{C}_0\left(\mathbb{R}^n\setminus\Omega_{j_0},F\right)$ *is K-null then for every clopen set* Γ' *of* Γ

$$K_i\left(\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\right)\approx K_{i+1}\left(\mathscr{C}_0\left(\Gamma',F\right)\right)\times K_{i+n}(F)\,.$$

a) follows from Proposition 2.4.10.

b) follows from a) and the six-term axiom (Axiom 1.2.7).

c) We use the notation of Proposition 2.3.7. For $\Gamma'=\Gamma$ the assertion follows from Proposition 2.3.7 c_2) and Theorem 3.2.2 b). Let

$$\tilde{\varphi}:\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n,F\right),$$

$$\tilde{\tilde{\varphi}}:\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma,F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)$$

be the inclusion maps,

$$\tilde{\psi}:\mathscr{C}_0\left(\mathbb{R}^n,F\right)\longrightarrow\mathscr{C}_0\left(\Gamma',F\right),\quad x\longmapsto x|\Gamma',$$

$\tilde{\delta}_i$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\xrightarrow{\tilde{\varphi}}\mathscr{C}_0\left(\mathbb{R}^n,F\right)\xrightarrow{\tilde{\psi}}\mathscr{C}_0\left(\Gamma',F\right)\longrightarrow 0,$$

and $\tilde{\Phi}_i:=K_i\left(\tilde{\tilde{\varphi}}\right)\circ\Phi_i$. Since $\varphi=\tilde{\varphi}\circ\tilde{\tilde{\varphi}}$,

$$K_i(\tilde{\varphi})\circ\tilde{\Phi}_i=K_i(\tilde{\varphi})\circ K_i\left(\tilde{\tilde{\varphi}}\right)\circ\Phi_i=K_i(\varphi)\circ\Phi_i=id_{K_i(\mathscr{C}_0(\mathbb{R}^n,F))}\,.$$

Thus

$$0\longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\Gamma',F\right)\right)\xrightarrow{\tilde{\delta}_{i+1}}K_i\left(\mathscr{C}_0\left(\mathbb{R}^n\setminus\Gamma',F\right)\right)\underset{\tilde{\Phi}_i}{\overset{K_i(\tilde{\varphi})}{\rightleftarrows}}K_i(\mathscr{C}_0\left(\mathbb{R}^n,F\right))\longrightarrow 0$$

is a split exact sequence and this implies c). ■

PROPOSITION 3.2.11 *Let Ω, Ω' be compact spaces and $m \in \mathbb{N}$. If Ω is path connected, $\Omega \times \Omega' \subset \mathbb{B}_n$, and $\mathbb{B}_n \setminus (\Omega \times \Omega')$ is homeomorphic to the topological sum of $\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m)$ and $\Omega \times (\mathbb{B}_m \setminus \Omega')$ then for all $\omega \in \Omega$ and $\omega_0 \in \Omega \times \Omega'$*

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

In particular if there is a $p \in \mathbb{N}$ such that $\mathbb{B}_m \setminus \Omega'$ is homeomorphic to p copies of $\mathbb{R}^m$ then

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p .$$

By Theorem 3.1.2 b) and the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \Omega'), F\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m), F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

By Theorem 3.1.2 b) and Corollary 3.1.5,

$$K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_n \setminus (\Omega \times \mathbb{B}_m), F\right)\right) \approx K_i\left(\mathscr{C}_0\left((\Omega \times \mathbb{B}_m) \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right)$$

and so

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) .$$

We prove now the last assertion. By Theorem 3.1.2 a),

$$K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^m, \mathscr{C}\left(\Omega, F\right)\right)\right) \approx K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)$$

so by the Product Theorem (Proposition 2.3.1 a)),

$$K_{i+1}\left(\mathscr{C}_0\left(\Omega \times (\mathbb{B}_m \setminus \Omega'), F\right)\right) \approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{B}_m \setminus \Omega', \mathscr{C}\left(\Omega, F\right)\right)\right) \approx$$

$$\approx K_{i+1}\left(\mathscr{C}_0\left(\mathbb{R}^m, \mathscr{C}\left(\Omega, F\right)\right)\right)^p \approx K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p ,$$

$$K_i\left(\mathscr{C}_0\left((\Omega \times \Omega') \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega\}, F\right)\right) \times K_{i+m+1}\left(\mathscr{C}\left(\Omega, F\right)\right)^p . \qquad \blacksquare$$

COROLLARY 3.2.12 *Let Ω be a connected graph contained in $\mathbb{B}_2$ and containing $\mathbb{S}_1$, r_0 and r_1 the number of vertices and chords of Ω, respectively, and Γ a nonempty finite subset of $\mathbb{S}_n \times \Omega$. Then*

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\,\Gamma}\ .$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbb{S}_n \times \Omega$. There is an embedding of $\mathbb{S}_n \times \Omega$ in $\mathbb{B}_{n+2}$ such that $\mathbb{B}_{n+2} \setminus (\mathbb{S}_n \times \Omega)$ is homeomorphic to the topological sum of $\mathbb{B}_{n+2} \setminus (\mathbb{S}_n \times \mathbb{B}_2)$ and $\mathbb{S}_n \times (\mathbb{B}_2 \setminus \Omega)$. Since $\mathbb{S}_n \times (\mathbb{B}_2 \setminus \Omega)$ is homeomorphic to $1 - r_0 + r_1$ copies of $\mathbb{S}_n \times \mathbb{R}^2$, we get by Proposition 3.2.11, for $\omega \in \mathbb{S}_n$,

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx$$

$$\approx K_i(\mathscr{C}_0(\mathbb{S}_n \setminus \{\omega\}, F)) \times K_{i+1}(\mathscr{C}(\mathbb{S}_n, F))^{1-r_0+r_1}\ .$$

By Theorem 3.2.2 a),b),

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{1-r_0+r_1}.$$

By Proposition 2.4.11,

$$K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_i(\mathscr{C}_0((\mathbb{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\,\Gamma - 1} \approx$$

$$\approx K_{i+n}(F) \times K_{i+1+n}(F)^{1-r_0+r_1} \times K_{i+1}(F)^{r_1-r_0+Card\,\Gamma}\ .$$

■

COROLLARY 3.2.13 *If*

$$\Omega := \mathbb{S}_{n-1} \cup \left(\bigcup_{j \in \mathbb{N}_n} \{\, \alpha \in \mathbb{B}_n \mid \alpha_j = 0 \,\} \right),$$

$m \in \mathbb{N}$, and Γ is a finite subset of $\mathbb{S}_m \times \Omega$ then

$$K_i(\mathscr{C}_0((\mathbb{S}_m \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{Card\,\Gamma - 1}\ .$$

Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \mathbf{S}_m \times \Omega$. There is an embedding of $\mathbf{S}_m \times \Omega$ in $\mathbb{B}_{m+n+1}$ such that $\mathbb{B}_{m+n+1} \setminus (\mathbf{S}_m \times \Omega)$ is homeomorphic to the topological sum of $\mathbb{B}_{m+n+1} \setminus (\mathbf{S}_m \times \mathbb{B}_n)$ and $\mathbf{S}_m \times (\mathbb{B}_n \setminus \Omega)$. Since $\mathbb{B}_n \setminus \Omega$ is homeomorphic to the topological sum of 2^n copies of $\mathbb{R}^n$, by Proposition 3.2.11, for $\omega \in \mathbf{S}_m$,

$$K_i(\mathscr{C}_0((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F)) \approx$$

$$\approx K_i(\mathscr{C}_0(\mathbf{S}_m \setminus \{\omega\}, F)) \times K_{i+n+1}(\mathscr{C}(\mathbf{S}_m, F))^{2^n} .$$

By Proposition 3.2.2 a),b),

$$K_i(\mathscr{C}_0((\mathbf{S}_n \times \Omega) \setminus \{\omega_0\}, F)) \approx K_{i+m}(F) \times K_{i+1+n}(F)^{2^n} \times K_{i+1+m+n}(F)^{2^n} .$$

By Proposition 2.4.11,

$$K_i(\mathscr{C}_0((\mathbf{S}_m \times \Omega) \setminus \Gamma, F)) \approx$$

$$\approx K_i(\mathscr{C}_0((\mathbf{S}_m \times \Omega) \setminus \{\omega_0\}, F)) \times K_{i+1}(F)^{Card\,\Gamma - 1} \approx$$

$$\approx K_{i+m}(F) \times K_{i+n+1}(F)^{2^n} \times K_{i+m+n+1}(F)^{2^n} \times K_{i+1}(F)^{Card\,\Gamma - 1} .$$ ■

LEMMA 3.2.14 *Let $(k_j)_{j\in\mathbb{N}_n}$ be a family in $\mathbb{N}$, $n \neq 1$, and $m := 1 + \sum_{j\in\mathbb{N}_n} k_j$. There is an embedding of $\prod_{j\in\mathbb{N}_n} \mathbf{S}_{k_j}$ in $\mathbb{B}_m$ such that $\mathbb{B}_m \setminus \prod_{j\in\mathbb{N}_n} \mathbf{S}_{k_j}$ has two connected components: one is homeomorphic to $\mathbb{R}^{1+k_n} \times \prod_{j\in\mathbb{N}_{n-1}} \mathbf{S}_{k_j}$ and the other is homeomorphic to $\mathbb{B}_m \setminus \left(\mathbb{B}_{1+k_n} \times \prod_{j\in\mathbb{N}_{n-1}} \mathbf{S}_{k_j}\right)$.*

We prove the assertion by induction with respect to $n \in \mathbb{N} \setminus \{1\}$. Assume first $n = 2$, put

$$\Gamma := \left\{ \alpha \in \mathbb{B}_m \mid \|\alpha\| = \frac{1}{2},\ \alpha_{2+k_1} = \alpha_{3+k_1} = \cdots = \alpha_m = 0 \right\},$$

and for every $\alpha \in \mathbb{B}_m$ denote by $d(\alpha)$ the distance of α to Γ. Then

$$\left\{ \alpha \in \mathbb{B}_m \mid d(\alpha) = \frac{1}{4} \right\}$$

is an embedding of $\mathbf{S}_{k_1} \times \mathbf{S}_{k_2}$ in $\mathbb{B}_m$ with the desired properties.

Let now $n > 2$ and assume the assertion holds for $n-1$. Let Γ be a closed set of $\mathbb{B}_{m-k_n}$ homeomorphic to $\prod_{j\in\mathbb{N}_{n-1}} \mathbb{S}_{k_j}$. We may assume $\Gamma \subset \mathbb{S}_{m-k_m}$. We denote for every $\alpha \in \mathbb{B}_m$ by $d(\alpha)$ the distance of α to $\frac{1}{2}\Gamma$. Then $\left\{ \alpha \in \mathbb{B}_m \mid d(\alpha) = \frac{1}{4} \right\}$ is an embedding with the desired properties. ■

PROPOSITION 3.2.15 *Let $(k_j)_{j\in\mathbb{N}_n}$ be a family in $\mathbb{N}$.*

a)
$$\prod_{j=1}^{n} \mathbb{S}_{k_j} \in \Upsilon, \qquad \mathbb{R}_\Upsilon \subset \left(\prod_{j=1}^{n} \mathbb{S}_{k_j}\right)_\Upsilon ,$$

$$K_i\left(\mathscr{C}\left(\prod_{j=1}^{n} \mathbb{S}_{k_j}, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^n} & \text{if} \quad \text{all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \\ \Big(K_i(F)\times K_{i+1}(F)\Big)^{2^{n-1}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \end{cases} .$$

b) If Γ is a nonempty finite subset of $\prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j}$ then

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathbb{N}_n} \mathbb{S}_{k_j}\setminus\Gamma, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^n-1}\times K_{i+1}(F)^{Card\,\Gamma-1} & \text{if} \quad \text{all } k_j \text{ are even} \\ K_i(F)^{2^{n-1}-1}\times K_{i+1}(F)^{2^{n-1}+Card\,\Gamma-2} & \text{if} \quad \text{not all } k_j \text{ are even} \end{cases} .$$

a) By Theorem 3.2.2 b), $\mathbb{S}_{k_j} \in \Upsilon, \mathbb{R}_\Upsilon \subset (\mathbb{S}_{k_j})_\Upsilon$ for every $j \in J$ so by Proposition 1.5.11 a),f), $\prod_{j=1}^{n} \mathbb{S}_{k_j} \in \Upsilon, \mathbb{R}_\Upsilon \subset \left(\prod_{j=1}^{n} \mathbb{S}_{k_j}\right)_\Upsilon$. By Theorem 3.2.2 b), with the notation of Proposition 1.5.11 a),f),

$$p_j = \frac{3+(-1)^{k_j}}{2}, \quad q_j = \frac{1-(-1)^{k_j}}{2}, \quad p_j+q_j = 2, \quad p_j-q_j = 1+(-1)^{k_j},$$

$$p_J = \frac{1}{2}\left(2^n + \prod_{j=1}^{n}\left(1+(-1)^{k_j}\right)\right), \quad q_J = \frac{1}{2}\left(2^n - \prod_{j=1}^{n}\left(1+(-1)^{k_j}\right)\right),$$

and this implies the result.

b) Assume first $\Gamma = \{\omega_0\}$ for some $\omega_0 \in \prod_{j\in\mathrm{I\!N_n}} \mathbf{S}_{k_j}$. We prove the assertion by induction with respect to $n \in \mathrm{I\!N}$. For $n = 1$ this follows from Theorem 3.2.2 e_1). Let $n \neq 1$ and assume the assertion holds for $n - 1$. By Lemma 3.2.14, $\mathrm{I\!B}_m \setminus \prod_{j\in\mathrm{I\!N_n}} \mathbf{S}_{k_j}$ is homeomorphic to the topological sum of $\mathrm{I\!R}^{1+k_n} \times \prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j}$ and $\mathrm{I\!B}_m \setminus \left(\mathrm{I\!B}_{1+k_n} \times \prod_{j\in\mathrm{I\!N_{n-i}}} \mathbf{S}_{k_j}\right)$. By Proposition 3.2.11, for $\omega \in \prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j}$,

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathrm{I\!N_n}} \mathbf{S}_{k_j} \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j} \setminus \{\omega\}, F\right)\right) \times$$

$$\times K_{i+1}\left(\mathscr{C}_0\left((\mathrm{I\!B}_m \setminus \mathbf{S}_{k_n}) \times \prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j}, F\right)\right).$$

By a) and Theorem 3.2.2 g),

$$K_{i+1}\left(\mathscr{C}_0\left((\mathrm{I\!B}_n \setminus \mathbf{S}_{k_n}) \times \prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j}, F\right)\right) \approx$$

$$\approx K_{i+k_n}\left(\mathscr{C}\left(\prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j}, F\right)\right) \approx$$

$$\approx \begin{cases} K_{i+k_n}(F)^{2^{n-1}} & \text{if} \quad \text{all } (k_j)_{j\in\mathrm{I\!N_{n-1}}} \text{ are even} \\ \left(K_i(F) \times K_{i+1}(F)\right)^{2^{n-2}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathrm{I\!N_{n-1}}} \text{ are even} \end{cases}.$$

By the induction hypothesis,

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathrm{I\!N_{n-1}}} \mathbf{S}_{k_j} \setminus \{\omega\}, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^{n-1}-1} & \text{if} \quad \text{all } (k_j)_{j\in\mathrm{I\!N_{n-1}}} \text{ are even} \\ K_i(F)^{2^{n-2}-1} \times K_{i+1}(F)^{2^{n-2}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathrm{I\!N_{n-1}}} \text{ are even} \end{cases}$$

so

$$K_i\left(\mathscr{C}_0\left(\prod_{j\in\mathrm{I\!N_n}} \mathbf{S}_{k_j} \setminus \{\omega_0\}, F\right)\right) \approx$$

$$\approx \begin{cases} K_i(F)^{2^n-1} & \text{if} \quad \text{all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \\ K_i(F)^{2^{n-1}-1} \times K_{i+1}(F)^{2^{n-1}} & \text{if} \quad \text{not all } (k_j)_{j\in\mathbb{N}_n} \text{ are even} \end{cases} .$$

This finishes the inductive proof.

We prove now the general case and put $\Omega := \prod_{j\in\mathbb{N}_n} \mathbf{S}_{k_j}$. Since it is possible to find a closed set Δ of Ω such that $\Gamma \subset \Delta$ and $\Delta \setminus \{\omega_0\}$ is K-null, the assertion follows from Proposition 2.4.11. ■

3.3 Some Morphisms

PROPOSITION 3.3.1 *We put*

$$\vartheta : \mathbb{B}_n \longrightarrow \mathbb{B}_n, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\vartheta' : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\vartheta'' : \mathbf{S}_{n-1} \longrightarrow \mathbf{S}_{n-1}, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto (\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n),$$

$$\phi : \mathscr{C}(\mathbb{B}_n, F) \longrightarrow \mathscr{C}(\mathbb{B}_n, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi' : \mathscr{C}_0(\mathbb{R}^n, F) \longrightarrow \mathscr{C}_0(\mathbb{R}^n, F), \quad x \longmapsto x \circ \vartheta',$$

$$\phi'' : \mathscr{C}(\mathbf{S}_{n-1}, F) \longrightarrow \mathscr{C}(\mathbf{S}_{n-1}, F), \quad x \longmapsto x \circ \vartheta''.$$

a) $K_i(\phi) : K_i(\mathscr{C}(\mathbb{B}_n, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{B}_n, F)), \quad a \longmapsto a.$

b) $K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{R}^n, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{R}^n, F)), \quad b \longmapsto -b.$

c)

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)),$$

$$(a,b) \longmapsto \begin{cases} (b,a) & \text{if} \quad n = 1 \\ (a,-b) & \text{if} \quad n > 1 \end{cases},$$

where we identified $K_i(\mathscr{C}(\mathbf{S}_{n-1}, F))$ *with*

$$K_i(\mathscr{C}(\mathbb{B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathbb{B}_n \setminus \mathbf{S}_{n-1}, F))$$

using the group isomorphism of Corollary 3.2.8 d) *if* $n > 1$.

a) follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathscr{C}(\mathrm{I\!B}_n, F)$.

b) We identify $\mathrm{I\!R}^n$ with the homeomorphic space $\mathrm{I\!B}_n \setminus \mathbf{S}_{n-1}$.

Assume first $n = 1$. Put

$$\psi : \mathscr{C}(\mathrm{I\!B}_1, F) \longrightarrow \mathscr{C}(\{-1,1\}, F)\,, \quad x \longmapsto x|\{-1,1\}$$

and denote by $\varphi : \mathscr{C}_0(]-1,1[, F) \longrightarrow \mathscr{C}(\mathrm{I\!B}_1, F)$ the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(]-1,1[, F) \xrightarrow{\varphi} \mathscr{C}(\mathrm{I\!B}_1, F) \xrightarrow{\psi} \mathscr{C}(\{-1,1\}, F) \longrightarrow 0\,.$$

By Corollary 2.4.7, $K_i(\psi)a = (a,a)$ for every $a \in K_i(\mathscr{C}(\mathrm{I\!B}_1, F))$ so by the six-term axiom (Axiom 1.2.7),

$$\delta_i(a+b, a+b) = 0\,, \qquad \delta_i(a,b) = -\delta_i(b,a)$$

for all $(a,b) \in K_i(\mathscr{C}(\{-1,1\}, F))$. By the commutativity of the index maps (Axiom 1.2.8), $K_{i+1}(\phi') \circ \delta_i = \delta_i \circ K_i(\phi'')$. For $(a,b) \in K_i(\mathscr{C}(\{-1,1\}, F))$, by the above,

$$K_{i+1}(\phi')\delta_i(a,b) = \delta_i K_i(\phi'')(a,b) = \delta_i(b,a) = -\delta_i(a,b)\,.$$

Since δ_i is surjective (because φ factorizes through null and is therefore K-null), $K_i(\phi')b = -b$ for all $b \in K_i(\mathscr{C}_0(]-1,1[, F))$.

If $n > 1$ then the assertion follows from the case $n = 1$, since $\mathscr{C}_0(\mathrm{I\!R}^n, F) \approx \mathscr{C}_0\left(\mathrm{I\!R}, \mathscr{C}_0\left(\mathrm{I\!R}^{n-1}, F\right)\right)$

c) follows from a), b), and Corollary 3.2.8 c). ■

COROLLARY 3.3.2 *If we put*

$$\vartheta : \mathrm{I\!B}_n \longrightarrow \mathrm{I\!B}_n\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\vartheta' : \mathrm{I\!R}^n \longrightarrow \mathrm{I\!R}^n\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\vartheta'' : \mathbf{S}_{n-1} \longrightarrow \mathbf{S}_{n-1}\,, \quad \alpha \longmapsto -\alpha\,,$$

$$\phi : \mathscr{C}(\mathrm{I\!B}_n, F) \longrightarrow \mathscr{C}(\mathrm{I\!B}_n, F)\,, \quad x \longmapsto x \circ \vartheta\,,$$

$$\phi' : \mathscr{C}_0(\mathrm{I\!R}^n, F) \longrightarrow \mathscr{C}_0(\mathrm{I\!R}^n, F)\,, \quad x \longmapsto x \circ \vartheta'\,,$$

$$\phi'' : \mathscr{C}(\mathbf{S}_{n-1}, F) \longrightarrow \mathscr{C}(\mathbf{S}_{n-1}, F)\,, \quad x \longmapsto x \circ \vartheta''$$

then

$$K_i(\phi) : K_i(\mathscr{C}(\mathrm{I\!B}_n, F)) \longrightarrow K_i(\mathscr{C}(\mathrm{I\!B}_n, F))\,, \quad a \longmapsto a\,,$$

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathrm{I\!R}^n, F)) \longrightarrow K_i(\mathscr{C}_0(\mathrm{I\!R}^n, F))\,, \quad b \longmapsto (-1)^n b\,,$$

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbf{S}_{n-1}, F)) \longrightarrow K_i(\mathscr{C}(\mathbf{S}_{n-1}, F))\,,$$

$$(a,b) \longmapsto \begin{cases} (b,a) & \text{if} \quad n = 1 \\ (a, (-1)^{n+1} b) & \text{if} \quad n > 1 \end{cases}\,,$$

where we identified $K_i(\mathscr{C}(\mathbf{S}_{n-1}, F))$ *with*

$$K_i(\mathscr{C}(\mathrm{I\!B}_n, F)) \times K_{i+1}(\mathscr{C}_0(\mathrm{I\!B}_n \setminus \mathbf{S}_{n-1}, F))$$

using the group isomorphism of Corollary 3.2.8 c) *if* $n > 1$.

The assertion for $K_i(\phi)$ follows from the homotopy axiom (Axiom 1.2.5) since ϕ is homotopic to the identity map of $\mathscr{C}(\mathrm{I\!B}_n, F)$. If n is even then the same holds for $K_i(\phi')$. Assume now n odd and let us denote by $\bar{\phi}'$ the map denoted by ϕ' in Proposition 3.3.1. Then $\phi' \circ \bar{\phi}'$ is homotopic to the identity map of $\mathscr{C}_0(\mathrm{I\!R}^n, F)$ so by Corollary 3.3.1, for every $b \in K_i(\mathscr{C}_0(\mathrm{I\!R}^n, F))$,

$$K_i(\phi')b = -K_i(\phi')K_i(\bar{\phi}')b = -b = (-1)^n b\,.$$

The assertion for $K_i(\phi'')$ follows from the corresponding assertions for $K_i(\phi)$ and $K_i(\phi')$ and from Corollary 3.2.8 d). ■

PROPOSITION 3.3.3 *Let* $\alpha, \beta \in [0, 2\pi[$, $\alpha < \beta$, $\Omega := \left\{ e^{i\omega} \;\middle|\; \omega \in]\alpha, \beta[\right\}$, $\Gamma := \mathbb{T} \setminus \Omega$, $\varphi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$ *the inclusion map, and*

$$\psi : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\Gamma, F)\,, \quad x \longmapsto x|\Gamma\,,$$

$$\bar{\psi} : \mathscr{C}(\Gamma, F) \longrightarrow F\,, \quad x \longmapsto x(1)\,,$$

$$\vartheta :]0, 2\pi[\longrightarrow]\alpha, \beta[\,, \quad \omega \longmapsto \frac{\beta - \alpha}{2\pi}\omega + \alpha\,.$$

For every $x \in \mathscr{C}_0(\Omega, F)$ *put*

$$\tilde{x} : \mathbb{T} \longrightarrow F, \quad e^{i\omega} \longmapsto \begin{cases} x\left(e^{i\vartheta(\omega)}\right) & \text{if} \quad \omega \in]0, 2\pi[\\ 0 & \text{if} \quad \omega \in \{0, 2\pi\} \end{cases}$$

and define

$$\phi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F), \quad x \longmapsto \tilde{x} .$$

a) $K_i(\phi)$ *and* $K_i(\bar{\psi})$ *are group isomorphisms and so*

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(F), \qquad K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F) .$$

b) If we identify $K_i(\mathscr{C}_0(\Omega, F))$ *with* $K_{i+1}(F)$ *and* $K_i(\mathscr{C}(\Gamma, F))$ *with* $K_i(F)$ *using the isomorphisms from a) and* $K_i(\mathscr{C}(\mathbb{T}, F))$ *with* $K_i(F) \times K_{i+1}(F)$ *using e.g. Alexandroff K-theorem* (Theorem 2.2.1 a)) *then*

$$K_i(\varphi) : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F)), \quad b \longmapsto (0, b),$$

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad (a, b) \longmapsto a .$$

a) ϕ is an E-C*-isomorphism. Put

$$\tilde{\psi} : F \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto 1_{\mathscr{C}(\Gamma, \mathbb{C})} x .$$

Then $\mathscr{C}(\Gamma, F) \xrightarrow{\bar{\psi}} F \xrightarrow{\tilde{\psi}} \mathscr{C}(\Gamma, F)$ is a homotopy in $\mathfrak{M}_E$ so $K_i(\phi)$ and $K_i(\bar{\psi})$ are group isomorphisms by the homotopy axiom (Axiom 1.2.5). The last assertion follows now from Theorem 3.2.2 a).

b) For every $s \in [0, 1]$ put

$$\vartheta_s : \mathbb{T} \longrightarrow \mathbb{T}, \quad e^{i\omega} \longmapsto \begin{cases} e^{is\omega} & \text{if} \quad \omega \in [0, \alpha] \\ e^{is\omega} e^{\frac{2\pi i(1-s)(\omega-\alpha)}{\beta-\alpha}} & \text{if} \quad \omega \in]\alpha, \beta[\\ e^{is\omega} e^{2\pi i(1-s)} & \text{if} \quad \omega \in [\beta, 2\pi] \end{cases},$$

$$\phi_s : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F), \quad x \longmapsto x \circ \vartheta_s .$$

Then $(\phi_s)_{s \in [0,1]}$ is a pointwise continuous path in $\mathscr{C}(\mathbb{T}, F)$ such that ϕ_1 is the identity map. By the homotopy axiom (Axiom 1.2.5), $K_i(\phi_0)$ is the identity map of $K_i(\mathscr{C}(\mathbb{T}, F))$. Let

$$\varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

be the inclusion map and

$$\psi' : \mathscr{C}(\mathbb{T}, F) \longrightarrow F\,, \quad x \longmapsto x(1)\,.$$

Then $\phi_0 \circ \varphi = \varphi' \circ \phi$ and $\psi' \circ \phi_0 = \bar{\psi} \circ \psi$ so (by a)) for $a \in K_i(F)$ and $b \in K_{i+1}(F)$,

$$K_i(\varphi)b = K_i(\phi_0)K_i(\varphi)b = K_i(\varphi')K_i(\phi)b = K_i(\varphi')b = (0, b)\,,$$

$$K_i(\psi)(a,b) = K_i(\bar{\psi})K_i(\psi)(a,b) = K_i(\psi')K_i(\phi_0)(a,b) = K_i(\psi')(a,b) = a$$

by the Alexandroff K-theorem (Theorem 2.2.1 a)). ■

PROPOSITION 3.3.4 *Put* $\Gamma := \left\{ e^{\frac{2\pi i j}{n}} \;\middle|\; j \in \mathbb{N}_n \right\}$ *and*

$$\psi : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\Gamma, F)\,, \quad x \longmapsto x|\Gamma\,,$$

and denote by

$$\varphi : \mathscr{C}_0(\mathbb{T} \setminus \Gamma, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

the inclusion map and by δ_i *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{T} \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}(\mathbb{T}, F) \xrightarrow{\psi} \mathscr{C}(\Gamma, F) \longrightarrow 0\,.$$

a) $K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(F)^n, \quad K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^n.$

b) We identify the isomorphic groups of a) and identify $K_i(\mathscr{C}(\mathbb{T}, F))$ *with* $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)).

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F))\,, \quad (b_j)_{j \in \mathbb{N}_n} \longmapsto \left(0, \sum_{j \in \mathbb{N}_n} b_j\right),$$

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F))\,, \quad (a,b) \longmapsto (a)_{j \in \mathbb{N}_n}\,.$$

If $n = 2$ *and* $K_i(F)$ *is isomorphic to* $\mathbf{Z}$ *or to* $\mathbf{Z}_p$ *for some* $p \in \mathbb{N}$ *or to the group of rational numbers then there is an automorphism*

$$\Phi_i : K_i(F) \longrightarrow K_i(F)$$

such that

$$\delta_i : K_i(\mathscr{C}(\Gamma, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F))\,, \quad (a,b) \longmapsto (\Phi_i(a-b), \Phi(b-a)).$$

c) If we put

$$\vartheta : \mathbb{T} \setminus \Gamma \longrightarrow \mathbb{T} \setminus \{1\}, \quad z \longmapsto z^n,$$

$$\vartheta' : \mathbb{T} \longrightarrow \mathbb{T}, \quad z \longmapsto z^n,$$

$$\vartheta'' : \Gamma \longrightarrow \{1\}, \quad z \longmapsto z^n,$$

$$\phi : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}_0(\mathbb{T} \setminus \Gamma, F), \quad x \longmapsto x \circ \vartheta,$$

$$\phi' : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F), \quad x \longmapsto x \circ \vartheta',$$

$$\phi'' : \mathscr{C}(\{1\}, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x \circ \vartheta''$$

then, with the identifications of a) and b),

$$K_i(\phi) : K_i(\mathscr{C}_0(\mathbb{T} \setminus \{1\}, F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)), \quad b \longmapsto (b)_{j \in \mathbb{N}_n},$$

$$K_i(\phi') : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F)), \quad (a,b) \longmapsto (a, nb),$$

$$K_i(\phi'') : K_i(\mathscr{C}(\{1\}, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad a \longmapsto (a)_{j \in \mathbb{N}_n}.$$

a) Put $\Omega_j := \left\{ e^{\frac{2\pi i \omega}{n}} \;\middle|\; \omega \in]j-1, j[\right\}$ for every $j \in \mathbb{N}_n$. By Proposition 3.3.3 a), for every $j \in \mathbb{N}_n$,

$$K_i(\mathscr{C}_0(\Omega_j, F)) \approx K_{i+1}(F).$$

so

$$K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)) \approx K_{i+1}(F)^n, \qquad K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F)^n$$

by the Product Theorem (Proposition 2.3.1 a)).

b) By Corollary 2.4.7,

$$K_i(\psi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad (a,b) \longmapsto (a)_{j \in \mathbb{N}_n}.$$

If we denote by

$$\varphi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

the inclusion map then

$$K_i(\varphi_j) : K_i(\mathscr{C}_0(\Omega_j, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F)), \quad b \longmapsto (0, b)$$

by Proposition 3.3.3 b). By Proposition 3.3.3 a) and Corollary 2.3.2,

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{T} \setminus \Gamma, F)) \longrightarrow K_i(\mathscr{C}(\mathbb{T}, F)), \quad (b_j)_{j \in \mathbb{N}_n} \longmapsto \left(0, \sum_{j \in \mathbb{N}_n} b_j\right).$$

In order to prove the last assertion we define $a', b', a'', b'' \in K_i(F)$ by

$$(a', b') := \delta_i(1,0), \qquad (a'', b'') := \delta_i(0,1) .$$

From

$$0 = \delta_i(1,1) = (a', b') + (a'', b'') = (a' + a'', b' + b'')$$

we get $a'' = -a'$ and $b'' = -b'$. There are $j, k \in \mathbf{Z}$ such that $\delta_i(j,k) = (1,-1)$. Then

$$(1,-1) = \delta_i(j,k) = (ja', jb') - (ka', kb') = ((j-k)a', (j-k)b'),$$

$$(j-k)a' = 1, \qquad (j-k)b' = -1 .$$

Thus a' is invertible in the ring $K_i(F)$ and $a'^{-1} = j - k$. It follows $b' = -a'$. If we put

$$\Phi_i : K_i(F) \longrightarrow K_i(F), \quad c \longmapsto a'c$$

then Φ_i is an automorphism and for all $a, b \in K_i(F)$,

$$\delta_i(a,b) = (a'a, -a'a) - (a'b, -a'b) = (a'(a-b), a'(b-a)) = (\Phi_i(a-b), \Phi_i(b-a)) .$$

c) The assertions for $K_i(\phi)$ and $K_i(\phi'')$ follow from the Product Theorem (Proposition 2.3.1 a)). If $\varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$ denotes the inclusion map and

$$\psi' : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}(\{1\}, F), \quad x \longmapsto x|\{1\}$$

then the diagram

$$\begin{array}{ccccc} K_i(\mathscr{C}_0(\mathbb{T}\setminus\{1\},F)) & \xrightarrow{K_i(\varphi')} & K_i(\mathscr{C}(\mathbb{T},F)) & \xrightarrow{K_i(\psi')} & K_i(\mathscr{C}(\{1\},F)) \\ {\scriptstyle K_i(\phi)}\downarrow & & \downarrow{\scriptstyle K_i(\phi')} & & \downarrow{\scriptstyle K_i(\phi'')} \\ K_i(\mathscr{C}_0(\mathbb{T}\setminus\Gamma,F)) & \xrightarrow[K_i(\varphi)]{} & K_i(\mathscr{C}(\mathbb{T},F)) & \xrightarrow[K_i(\psi)]{} & K_i(\mathscr{C}(\Gamma,F)) \end{array}$$

is commutative. Let $(a,b) \in K_i(\mathscr{C}(\mathbb{T}, F))$ and put $(a', b') := K_i(\phi')(a, 0)$. By b),

$$(a)_{j \in \mathbb{N}_n} = K_i(\phi'')a = K_i(\phi'')K_i(\psi')(a,0) =$$

$$= K_i(\psi)K_i(\phi')(a,0) = K_i(\psi)(a', b') = (a')_{j \in \mathbb{N}_n},$$

$$K_i(\phi')(0,b) = K_i(\phi')K_i(\varphi')b = K_i(\varphi)K_i(\phi)b = K_i(\varphi)(b)_{j \in \mathbb{N}_n} = (0, nb)$$

so $K_i(\phi')(a,b) = (a, nb)$. ■

COROLLARY 3.3.5 *If we put*

$$\vartheta : \mathbb{B}_2 \longrightarrow \mathbb{B}_2\,, \quad z \longmapsto z^n\,,$$

$$\vartheta' : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad z \longmapsto z^n\,,$$

$$\vartheta'' : \mathbb{S}_1 \longrightarrow \mathbb{S}_1\,, \quad z \longmapsto z^n\,,$$

$$\phi : \mathscr{C}(\mathbb{B}_2,F) \longrightarrow \mathscr{C}(\mathbb{B}_2,F)\,, \quad x \longmapsto x\circ\vartheta\,,$$

$$\phi' : \mathscr{C}_0(\mathbb{C},F) \longrightarrow \mathscr{C}_0(\mathbb{C},F)\,, \quad x \longmapsto x\circ\vartheta'\,,$$

$$\phi'' : \mathscr{C}(\mathbb{S}_1,F) \longrightarrow \mathscr{C}_0(\mathbb{S}_1,F)\,, \quad x \longmapsto x\circ\vartheta''\,.$$

then $K_i(\phi)$ is the identity map of $K_i(\mathscr{C}(\mathbb{B}_2,F))$ and

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{C},F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{C},F))\,, \quad a \longmapsto na\,,$$

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbb{S}_1,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_1,F))\,, \quad (a,b) \longmapsto (a,nb)\,,$$

We identify the homeomorphic spaces $\mathbb{C}$ and $\mathbb{B}_2\setminus\mathbb{S}_1$. By Corollary 3.2.8 c),

$$K_i(\mathscr{C}(\mathbb{S}_1,F)) \approx K_i(\mathscr{C}(\mathbb{B}_2,F))\times K_{i+1}(\mathscr{C}_0(\mathbb{B}_2\setminus\mathbb{S}_1,F))$$

and by Proposition 3.3.4 e),

$$K_i(\phi'') : K_i(\mathscr{C}(\mathbb{S}_1,F)) \longrightarrow K_i(\mathscr{C}(\mathbb{S}_1,F))\,, \quad (a,b) \longmapsto (a,nb)\,.$$

By Corollary 2.2.2 b) and Theorem 3.1.2 a), $K_i(\phi)$ is the identity map of $K_i(\mathscr{C}(\mathbb{B}_2,F))$ and

$$K_i(\phi') : K_i(\mathscr{C}_0(\mathbb{B}_2\setminus\mathbb{S}_1,F)) \longrightarrow K_i(\mathscr{C}_0(\mathbb{B}_2\setminus\mathbb{S}_1,F))\,, \quad a \longmapsto na\,. \qquad \blacksquare$$

PROPOSITION 3.3.6 *Let $m,n\in\mathbb{N}$ and*

$$\vartheta_1 : \mathbb{T} \longrightarrow \mathbb{T}\,, \quad w \longmapsto w^m\,,$$

$$\vartheta_2 : \mathbb{T} \longrightarrow \mathbb{T}\,, \quad z \longmapsto z^n\,,$$

$$\psi : \mathscr{C}(\mathbb{T}\times\mathbb{T},F) \longrightarrow \mathscr{C}(\mathbb{T}\times\mathbb{T},F)\,, \quad x \longmapsto x\circ(\vartheta_1\times\vartheta_2)\,.$$

We identify $K_i(\mathscr{C}(\mathbb{T},F'))$ with $K_i(F')\times K_{i+1}(F')$ for all E-C-algebras F' by using the group isomorphism of* Theorem 3.2.2 b). *Let*

$$a\in K_i(\mathscr{C}(\mathbb{T}\times\mathbb{T},F)) \approx K_i(\mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))) \approx K_i(\mathscr{C}(\mathbb{T},F))\times K_{i+1}(\mathscr{C}(\mathbb{T},F))$$

and put $a_0 \in K_i(\mathscr{C}(\mathbb{T},F))$, $a_1 \in K_{i+1}(\mathscr{C}(\mathbb{T},F))$ such that $a=(a_0,a_1)$ and $a_{0,0}, a_{1,1} \in K_i(F)$ and $a_{0,1}, a_{1,0} \in K_{i+1}(F)$ such that $a_0=(a_{0,0}, a_{0,1})$ and $a_1=(a_{1,0}, a_{1,1})$. Then

$$K_i(\psi) = ((a_{0,0}, mna_{1,1}), (na_{0,1}, ma_{1,0}))\ .$$

We put

$$\phi : \mathscr{C}(\mathbb{T},F) \longrightarrow \mathscr{C}(\mathbb{T},F)\,, \quad x \longmapsto x \circ \vartheta_2\,,$$

$$\phi_1 : \mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F)) \longrightarrow \mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\,, \quad x \longmapsto x \circ \vartheta_1\,,$$

$$\phi_2 : \mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F)) \longrightarrow \mathscr{C}(\mathbb{T},\mathscr{C}(\mathbb{T},F))\,, \quad x \longmapsto \phi \circ x\,,$$

By Corollary 3.3.5,

$$K_i(\phi_1)a = (a_0, ma_1), \quad K_i(\phi)a_0 = (a_{0,0}, na_{0,1}), \quad K_{i+1}(\phi)a_1 = (a_{1,0}, na_{1,1})$$

so by Corollary 3.2.8 c), d),

$$K_i(\phi_2)K_i(\phi_1)a = (K_i(\phi)a_0, K_{i+1}(\phi)ma_1) = ((a_{0,0}, na_{0,1}), (ma_{1,0}, mna_{1,1}))\ .$$

Since $\psi = \phi_2 \circ \phi_1$,

$$K_i(\psi) = ((a_{0,0}, mna_{1,1}), (na_{0,1}, ma_{1,0}))\ . \qquad \blacksquare$$

3.4 Some Non-orientable Compact Spaces

DEFINITION 3.4.1 *We denote by $\mathbb{P}_n$ the n-dimensional* **projective space***, which is obtained from $\mathbb{B}_n$ by identifying α with $-\alpha$ for all $\alpha \in \mathbb{B}_n$ with $\|\alpha\| = 1$.*

PROPOSITION 3.4.2 *Put*

$$\Omega := \mathbb{P}_{n+1} \setminus \{\, \alpha \in \mathbb{P}_{n+1} \mid \|\alpha\| = 1, \alpha_{n+1} = 0 \,\}\ ,$$

$$\Omega' := \{\, \alpha \in \Omega \mid \|\alpha\| = 1 \,\}\ ,$$

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Omega', F)\,, \quad x \longmapsto x|\Omega'$$

and denote by

$$\varphi : \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \longrightarrow \mathscr{C}_0(\Omega, F)$$

the inclusion map and by δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_{n+1} \setminus \mathbf{S}_n, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Omega', F) \longrightarrow 0\ .$$

a) $K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)) \approx K_{i+n+1}(F)$, $K_i(\mathscr{C}_0(\Omega',F)) \approx K_{i+n}(F)$, *and there is an automorphism* $\Phi_i : K_{i+n}(F) \longrightarrow K_{i+n}(F)$ *such that*

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)),$$

$$a \longmapsto \Phi_i(a-(-1)^n a)\ .$$

b) If n is even then $\delta_i = 0$, $K_i(\varphi)$ *is injective,* $K_i(\psi)$ *is surjective, and*

$$\frac{K_i(\mathscr{C}_0(\Omega,F))}{K_{i+1}(F)} \approx K_i(F)\ .$$

c) If n is odd and for a fixed $i\in\{0,1\}$

$$a\in K_i(F),\, 2a=0 \Longrightarrow a=0$$

then $K_i(\psi)=0$, $K_i(\mathscr{C}_0(\Omega,F)) \approx \frac{K_i(F)}{2K_i(F)}$,

$$K_i(\varphi) : K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)) \longrightarrow K_i(\mathscr{C}_0(\Omega,F))$$

is the quotient map, and

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)), \quad a \longmapsto 2\Phi_i a\ .$$

a) By Theorem 3.2.2 a), $K_i(\mathscr{C}_0(\mathbb{R}^n,F)) \approx K_{i+n}(F)$. Since $\mathbb{B}_{n+1}\setminus\mathbf{S}_n$ is homeomorphic to $\mathbb{R}^{n+1}$, $K_i(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)) \approx K_{i+n+1}(F)$. Since Ω' is homeomorphic to $\mathbb{R}^n$, $K_i(\mathscr{C}_0(\Omega',F)) \approx K_{i+n}(F)$. We use the notation of Proposition 3.2.9, which we mark by a bar in order to distinguish it from the present notation. Moreover we denote by $\vartheta : \bar{\Omega} \longrightarrow \Omega$ and $\vartheta' : \bar{\Omega}' \longrightarrow \Omega'$ the covering maps and put

$$\phi : \mathscr{C}_0(\Omega,F) \longrightarrow \mathscr{C}_0\left(\bar{\Omega},F\right), \quad x \longmapsto x\circ\vartheta\,,$$

$$\phi' : \mathscr{C}_0\left(\Omega',F\right) \longrightarrow \mathscr{C}_0\left(\bar{\Omega}',F\right), \quad x \longmapsto x\circ\vartheta'\ .$$

By the Product Theorem (Proposition 2.3.1 a)), Proposition 3.2.9 a), and Proposition 3.3.1 b),

$$K_i(\phi') : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_i\left(\mathscr{C}_0\left(\bar{\Omega}',F\right)\right), \quad a \longmapsto (a,(-1)^n a)\ .$$

By the commutativity of the index maps (Axiom 1.2.8), $\delta_i = \bar{\delta}_i \circ K_i(\phi')$ so by Proposition 3.2.9 b),

$$\delta_i : K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_{n+1}\setminus\mathbf{S}_n,F)), \quad a \longmapsto \Phi_i(a-(-1)^n a)\ .$$

b) and c) follow from a) and the six-term axiom (Axiom 1.2.7). ■

COROLLARY 3.4.3 *We use the notation and the hypothesis of* Proposition 3.4.2*, take* $n = 1$*, put* $\Gamma := \{ x \in \mathbb{P}_2 \mid \|x\| = 1 \}$*,*

$$\psi' : \mathscr{C}(\mathbb{P}_2, F) \longrightarrow \mathscr{C}(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and denote by $\varphi' : \mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F) \longrightarrow \mathscr{C}(\mathbb{P}_2, F)$ *the inclusion map and by* δ_i' *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F) \xrightarrow{\varphi'} \mathscr{C}(\mathbb{P}_2, F) \xrightarrow{\psi'} \mathscr{C}(\Gamma, F) \longrightarrow 0 .$$

Then $K_i(\mathscr{C}(\mathbb{P}_2, F)) \approx K_i(F) \times \frac{K_i(F)}{2K_i(F)}$*,* $K_i(\mathscr{C}(\Gamma, F)) \approx K_i(F) \times K_{i+1}(F)$*,*

$$K_i(\varphi') : K_i(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F)) \longrightarrow K_i(K_i(\mathscr{C}(\mathbb{P}, F))), \quad a \longmapsto (0, \Phi_i a),$$

$$K_i(\psi') : K_i(\mathscr{C}(\mathbb{P}_2, F)) \longrightarrow K_i(\mathscr{C}(\Gamma, F)), \quad (a, c) \longmapsto (a, 0),$$

$$\delta_i' : K_i(\mathscr{C}(\Gamma, F)) \longrightarrow K_{i+1}(\mathscr{C}_0(\mathbb{B}_2 \setminus \mathbb{S}_1, F)), \quad (a, b) \longmapsto 2b . \qquad \blacksquare$$

PROPOSITION 3.4.4 *Let*

$$\vartheta : [0,1] \longrightarrow \mathbb{T}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi : \mathscr{C}(\mathbb{T}, F) \longrightarrow \mathscr{C}([0,1], F), \quad x \longmapsto x \circ \vartheta .$$

If we identify $K_i(\mathscr{C}(\mathbb{T}, F))$ *with* $K_i(F) \times K_{i+1}(F)$ (Theorem 3.2.2 b)) *and* $K_i(\mathscr{C}([0,1], F))$ *with* $K_i(F)$ (Theorem 3.1.2 a)) *then*

$$K_i(\phi) : K_i(\mathscr{C}(\mathbb{T}, F)) \longrightarrow K_i(\mathscr{C}([0,1], F)), \quad (a, b) \longmapsto a .$$

Put

$$\vartheta' :]0,1[\longrightarrow \mathbb{T} \setminus \{1\}, \quad \omega \longmapsto e^{2\pi i \omega},$$

$$\phi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}_0(]0,1[, F), \quad x \longmapsto x \circ \vartheta'$$

and denote by

$$\varphi : \mathscr{C}_0(]0,1[, F) \longrightarrow \mathscr{C}([0,1], F), \quad \varphi' : \mathscr{C}_0(\mathbb{T} \setminus \{1\}, F) \longrightarrow \mathscr{C}(\mathbb{T}, F)$$

the inclusion maps. Then $\phi \circ \varphi' = \varphi \circ \phi'$, so $K_i(\phi) \circ K_i(\varphi') = K_i(\varphi) \circ K_i(\phi') = 0$, since φ factorizes through 0. Thus $K_i(\phi)(0, b) = 0$ for all $b \in K_{i+1}(F)$.

Put

$$\psi : \mathscr{C}([0,1],F) \longrightarrow \mathscr{C}(\{0,1\},F) \approx F \times F\,, \quad x \longmapsto x|\{0,1\}\,,$$

$$\psi' : \mathscr{C}(\mathbb{T},F) \longrightarrow F\,, \quad x \longmapsto x(1)\,,$$

$$\mu : F \longrightarrow \mathscr{C}(\{0,1\},F)\,, \quad x \longmapsto (x,x)\,.$$

Then $\psi \circ \phi = \mu \circ \psi'$, so $K_i(\psi) \circ K_i(\phi) = K_i(\mu) \circ K_i(\psi')$ and we get (by the above)

$$K_i(\psi)K_i(\phi)(a,b) = K_i(\psi)K_i(\phi)(a,0) = K_i(\mu)K_i(\psi')(a,0) = K_i(\mu)a = (a,a)\,,$$

$$K_i(\phi)(a,b) = a$$

for all $(a,b) \in K_i(F) \times K_{i+1}(F)$. ■

DEFINITION 3.4.5 *We denote by* $\mathbb{M}$ *the* **Möbius band** *obtained from* $[0,1] \times [-1,1]$ *by identifying the points* $(0,\beta)$ *and* $(1,-\beta)$ *for every* $\beta \in [-1,1]$. *We put for every* $j \in \{-1,0,1\}$

$$\Gamma_j^{\mathbb{M}} := \{\, (\alpha, j) \in \mathbb{M} \mid \alpha \in [0,1] \,\}\,.$$

PROPOSITION 3.4.6 *For every* $j \in \{-1,0,1\}$ *put*

$$\psi_j : \mathscr{C}(\mathbb{M},F) \longrightarrow \mathscr{C}\left(\Gamma_j^{\mathbb{M}},F\right)\,, \quad x \longmapsto x|\Gamma_j^{\mathbb{M}}\,.$$

a) $\Gamma_0^{\mathbb{M}}$ *is homeomorphic to* $\mathbb{T}$ *and* $\Gamma_j^{\mathbb{M}}$ *is homeomorphic to* $[0,1]$ *for all* $j \in \{-1,1\}$.

b) $\mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}},F\right)$ *is K-null and*

$$K_i(\psi_0) : K_i(\mathscr{C}(\mathbb{M},F)) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right)\right) \approx K_i(F) \times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathscr{C}(\mathbb{M},F))$ *with* $K_i(F) \times K_{i+1}(F)$ *using the group isomorphism* $K_i(\psi_0)$ *of b) and* $K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\right)$ *with* $K_i(F)$ *using a)* (and Theorem 3.1.2 a)) *then*

$$K_i(\psi_1) : K_i(\mathscr{C}(\mathbb{M},F)) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\right)\,, \quad (a,b) \longmapsto a\,.$$

d) If we put $\omega := (0,0) = (1,0) \in \mathbb{M}$, $\Gamma := \{\, (\alpha,0) \mid \alpha \in]0,1[\,\}$, *and*

$$\psi : \mathscr{C}_0(\mathbb{M} \setminus \{\omega\},F) \longrightarrow \mathscr{C}_0(\Gamma,F)\,, \quad x \longmapsto x|\Gamma$$

then

$$K_i(\psi) : K_i(\mathscr{C}_0(\mathbb{M} \setminus \{\omega\},F)) \longrightarrow K_i(\mathscr{C}_0(\Gamma,F)) \approx K_{i+1}(F)$$

is a group isomorphism.

e) If Γ' is a finite subset of $\mathbb{M}$ then

$$K_i\left(\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma',F\right)\right)\approx K_{i+1}(F)^{\Gamma'}$$

a) is easy to see.

b) For every $s\in]0,1]$ put

$$\vartheta_s:\mathbb{M}\setminus\Gamma_0^{\mathbb{M}}\longrightarrow\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},\quad(\alpha,\beta)\longmapsto(\alpha,s\beta)\,.$$

By Proposition 2.4.1 (replacing there Ω by $\mathbb{M}\setminus\Gamma_0^{\mathbb{M}}$), $\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)$ is K-null and the assertion follows from the Topological six-term sequence (Proposition 2.1.8 a)) and a) (and Theorem 3.2.2 b)).

c) follows from b) and Proposition 3.4.4.

d) If $\varphi:\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\{\omega\},F\right)$ denotes the inclusion map then

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)\xrightarrow{\varphi}\mathscr{C}_0\left(\mathbb{M}\setminus\{\omega\},F\right)\xrightarrow{\psi}\mathscr{C}_0\left(\Gamma,F\right)\longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. By b), $\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right)$ is K-null so by the Topological six-term sequence (Proposition 2.1.8 a)), $K_i(\psi)$ is a group isomorphism. Since Γ is homeomorphic to $\mathbb{R}$, $K_i(\mathscr{C}_0\left(\Gamma,F\right))\approx K_{i+1}(F)$ by Theorem 3.2.2 a).

e) follows from d) and Proposition 2.4.11. ■

PROPOSITION 3.4.7 *Put*

$$\Gamma':=\Gamma_0^{\mathbb{M}}\cup\Gamma_1^{\mathbb{M}},\qquad\Gamma'':=\Gamma_0^{\mathbb{M}}\cup\Gamma_1^{\mathbb{M}}\cup\Gamma_{-1}^{\mathbb{M}},\qquad\Gamma''':=\Gamma_1^{\mathbb{M}}\cup\Gamma_{-1}^{\mathbb{M}},$$

$$\mathbb{M}':=\mathbb{M}\setminus\Gamma',\qquad\mathbb{M}'':=\mathbb{M}\setminus\Gamma''\qquad\mathbb{M}''':=\mathbb{M}\setminus\Gamma'''\,.$$

Let

$$\varphi':\mathscr{C}_0\left(\mathbb{M}',F\right)\longrightarrow\mathscr{C}\left(\mathbb{M},F\right),$$

$$\varphi'':\mathscr{C}_0\left(\mathbb{M}'',F\right)\longrightarrow\mathscr{C}\left(\mathbb{M},F\right),$$

$$\bar{\varphi}':\mathscr{C}_0\left(\mathbb{M}',F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right),$$

$$\bar{\varphi}'':\mathscr{C}_0\left(\mathbb{M}'',F\right)\longrightarrow\mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right),$$

$$\varphi''' : \mathscr{C}_0\left(\mathbb{M}'',F\right) \longrightarrow \mathscr{C}_0\left(\mathbb{M}''',F\right)\,,$$

$$\lambda' : \mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right) \longrightarrow \mathscr{C}\left(\Gamma',F\right)\,,$$

$$\lambda'' : \mathscr{C}\left(\Gamma''',F\right) \longrightarrow \mathscr{C}\left(\Gamma'',F\right)$$

$$\lambda''' : \mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right) \longrightarrow \mathscr{C}\left(\Gamma'',F\right)\,,$$

be the inclusion maps,

$$\psi' : \mathscr{C}\left(\mathbb{M},F\right) \longrightarrow \mathscr{C}\left(\Gamma',F\right)\,, \quad x \longmapsto x|\Gamma'\,,$$

$$\psi'' : \mathscr{C}\left(\mathbb{M},F\right) \longrightarrow \mathscr{C}\left(\Gamma'',F\right)\,, \quad x \longmapsto x|\Gamma''\,,$$

$$\bar{\psi}' : \mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right) \longrightarrow \mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\,, \quad x \longmapsto x|\Gamma_1^{\mathbb{M}}\,,$$

$$\bar{\psi}'' : \mathscr{C}_0\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right) \longrightarrow \mathscr{C}\left(\Gamma''',F\right)\,, \quad x \longmapsto x|\Gamma'''\,,$$

$$\psi''' : \mathscr{C}_0\left(\mathbb{M}''',F\right) \longrightarrow \mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right)\,, \quad x \longmapsto x|\Gamma_0^{\mathbb{M}}\,,$$

and $\delta_i'\,,\delta_i''\,,\bar{\delta}_i'\,,\bar{\delta}_i''\,,\delta_i'''$ *the index maps associated to the exact sequences in* $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}',F\right) \xrightarrow{\varphi'} \mathscr{C}\left(\mathbb{M},F\right) \xrightarrow{\psi'} \mathscr{C}\left(\Gamma',F\right) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'',F\right) \xrightarrow{\varphi''} \mathscr{C}\left(\mathbb{M},F\right) \xrightarrow{\psi''} \mathscr{C}\left(\Gamma'',F\right) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}',F\right) \xrightarrow{\bar{\varphi}'} \mathscr{C}\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right) \xrightarrow{\bar{\psi}'} \mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'',F\right) \xrightarrow{\bar{\varphi}''} \mathscr{C}\left(\mathbb{M}\setminus\Gamma_0^{\mathbb{M}},F\right) \xrightarrow{\bar{\psi}''} \mathscr{C}\left(\Gamma''',F\right) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathscr{C}_0\left(\mathbb{M}'',F\right) \xrightarrow{\varphi'''} \mathscr{C}_0\left(\mathbb{M}''',F\right) \xrightarrow{\psi'''} \mathscr{C}\left(\Gamma_0^{\mathbb{M}},F\right) \longrightarrow 0\,,$$

respectively.

a) Γ''' *is homeomorphic to* $\mathbb{T}$.

b) The maps

$$\bar{\delta}_i' : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{M}},F\right)\right) \approx K_i(F) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}',F\right)\right)\,,$$

$$\bar{\delta}_i'' : K_i\left(\mathscr{C}\left(\Gamma''',F\right)\right) \approx K_i(F)\times K_{i+1}(F) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right)$$

are group isomorphisms.

c) *If we put* $\Phi'_i := K_i(\lambda') \circ (\bar{\delta}'_i)^{-1}$, $\Phi''_i := K_i(\lambda'') \circ (\bar{\delta}''_i)^{-1}$ *(using b)) then the sequences*

$$0 \longrightarrow K_i(\mathscr{C}(\mathrm{I\!M}, F)) \xrightarrow{K_i(\psi')} K_i\left(\mathscr{C}\left(\Gamma', F\right)\right) \underset{\Phi'_i}{\overset{\delta'_i}{\rightleftarrows}} K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!M}', F\right)\right) \longrightarrow 0\,,$$

$$0 \longrightarrow K_i(\mathscr{C}(\mathrm{I\!M}, F)) \xrightarrow{K_i(\psi'')} K_i\left(\mathscr{C}\left(\Gamma'', F\right)\right) \underset{\Phi''_i}{\overset{\delta''_i}{\rightleftarrows}} K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!M}'', F\right)\right) \longrightarrow 0$$

are split exact and the maps

$$K_i(\mathscr{C}(\mathrm{I\!M}, F)) \times K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!M}', F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma', F\right)\right),$$

$$(a,b) \longmapsto K_i(\psi')a + \Phi'_i b\,,$$

$$K_i(\mathscr{C}(\mathrm{I\!M}, F)) \times K_{i+1}\left(\mathscr{C}_0\left(\mathrm{I\!M}'', F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma'', F\right)\right),$$

$$(a,b) \longmapsto K_i(\psi'')a + \Phi''_i b$$

are group isomorphisms.

d) $\delta'''_i = 0$ *and the sequence*

$$0 \longrightarrow K_i\left(\mathscr{C}_0\left(\mathrm{I\!M}'', F\right)\right) \overset{K_i(\varphi''')}{\longrightarrow} K_i\left(\mathscr{C}_0\left(\mathrm{I\!M}''', F\right)\right)$$

$$K_i\left(\mathscr{C}_0\left(\mathrm{I\!M}''', F\right)\right) \overset{K_i(\psi''')}{\longrightarrow} K_i\left(\mathscr{C}\left(\Gamma_0^{\mathrm{I\!M}}, F\right)\right) \longrightarrow 0$$

is exact.

a) is easy to see.

b) By Proposition 3.4.6 b), $\mathscr{C}_0\left(\mathrm{I\!M} \setminus \Gamma_0^{\mathrm{I\!M}}, F\right)$ is K-null and the assertion follows from a), the Topological six-term sequence (Proposition 2.1.8 b)), and Proposition 3.4.6 a) (and Theorem 3.1.2 a), Theorem 3.2.2 b)).

c) If we put $\Omega_1 := \mathrm{I\!M}$, $\Omega_2 := \mathrm{I\!M} \setminus \Gamma_0^{\mathrm{I\!M}}$, and $\Omega_3 := \mathrm{I\!M}'$ (respectively $\Omega_3 := \mathrm{I\!M}''$) then the assertion follows from the Topological triple (Proposition 2.1.11 a)).

d) By the commutativity of the index maps (Axiom 1.2.8), $\delta'''_i = \delta''_i \circ K_i(\lambda''')$. By c), $Im\,(\Phi''_i \circ \delta''_i) \subset Im\,K_i(\lambda'')$. Since $Im\,K_i(\lambda''') = K_i\left(\mathscr{C}\left(\Gamma_0^{\mathrm{I\!M}}, F\right)\right)$ we get

$$\Phi''_i \circ \delta'''_i = \Phi''_i \circ \delta''_i \circ K_i(\lambda''') = 0\,.$$

Thus $\delta'''_i = \delta''_i \circ \Phi''_i \circ \delta'''_i = 0$ and the assertion follows from the six-term axiom (Axiom 1.2.7). ∎

DEFINITION 3.4.8 *We denote by* $\mathbb{K}$ *the* **Klein bottle** *obtained from the Möbius band* $\mathbb{M}$ *by identifying the points* $(\alpha,-1)$ *and* $(\alpha,1)$ *for all* $\alpha\in[0,1]$ *and put for every* $j\in\{0,1\}$

$$\Gamma_j^{\mathbb{K}} := \{\ (\alpha,j)\in\mathbb{K}\ |\ \alpha\in[0,1]\,\}\ .$$

PROPOSITION 3.4.9 *We put* $\mathbb{K}' := \mathbb{K}\setminus\Gamma_0^{\mathbb{K}}$, $\mathbb{K}'' := \mathbb{K}\setminus(\Gamma_0^{\mathbb{K}}\cup\Gamma_1^{\mathbb{K}})$,

$$\psi : \mathscr{C}_0\left(\mathbb{K}',F\right) \longrightarrow \mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\,,\quad x\longmapsto x|\Gamma_1^{\mathbb{K}}$$

and denote by $\varphi:\mathscr{C}_0(\mathbb{K}'',F)\longrightarrow\mathscr{C}_0(\mathbb{K}',F)$ *the inclusion map and by* δ_i *the index maps associated to the exact sequence in* $\mathfrak{M}_E$

$$0\longrightarrow\mathscr{C}_0\left(\mathbb{K}'',F\right)\xrightarrow{\ \varphi\ }\mathscr{C}_0\left(\mathbb{K}',F\right)\xrightarrow{\ \psi\ }\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\longrightarrow 0\,.$$

We use the notation of Proposition 3.4.7 *(so* $\Gamma_0^{\mathbb{K}}=\Gamma_0^{\mathbb{M}}$ *and* $\mathbb{K}''=\mathbb{M}''$*).*

a) $\Gamma_0^{\mathbb{K}}$ *and* $\Gamma_1^{\mathbb{K}}$ *are homeomorphic to* $\mathbb{T}$.

b) The map

$$(\bar{\delta}''_{i+1})^{-1} : K_i\left(\mathscr{C}_0\left(\mathbb{M}'',F\right)\right)\longrightarrow K_{i+1}\left(\mathscr{C}\left(\Gamma''',F\right)\right)\approx K_i(F)\times K_{i+1}(F)$$

is a group isomorphism.

c) If we identify $K_i(\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right))$ *with* $K_i(F)\times K_{i+1}(F)$ *using a) and* Theorem 3.2.2 b) *and* $K_{i+1}(\mathscr{C}_0(\mathbb{K}'',F))$ *with* $K_i(F)\times K_{i+1}(F)$ *using b) then*

$$\delta_i : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}},F\right)\right)\longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{K}'',F\right)\right)\,,\quad (a,b)\longmapsto(a,2b)\,.$$

d) If δ_i *is injective then* ψ *is K-null and* $K_i(\mathscr{C}_0(\mathbb{K}',F))\approx\frac{K_i(F)}{2K_i(F)}$ *and if we denote by*

$$\Phi_i : K_i(F)\longrightarrow\frac{K_i(F)}{2K_i(F)}$$

the quotient map then

$$K_i(\varphi) : K_i\left(\mathscr{C}_0\left(\mathbb{K}'',F\right)\right)\longrightarrow K_i\left(\mathscr{C}_0\left(\mathbb{K}',F\right)\right)\,,\quad (a,b)\longmapsto\Phi_i b\,.$$

a) is easy to see.

b) follows from Proposition 3.4.7 b).

c) We denote by

$$\vartheta : \mathbb{M} \setminus \Gamma_0^{\mathbb{M}} \longrightarrow \mathbb{K}'$$

the covering map, by

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}$$

the map defined by ϑ, and put

$$\phi : \mathscr{C}_0\left(\mathbb{K}', F\right) \longrightarrow \mathscr{C}_0\left(\mathbb{M} \setminus \Gamma_0^{\mathbb{M}}, F\right), \quad x \longmapsto x \circ \vartheta,$$

$$\phi' : \mathscr{C}_0\left(\Gamma_1^{\mathbb{K}}, F\right) \longrightarrow \mathscr{C}_0\left(\Gamma''', F\right), \quad x \longmapsto x \circ \vartheta'.$$

With the identifications of Γ''' and $\Gamma_1^{\mathbb{K}}$ with $\mathbb{T}$ (by a) and Proposition 3.4.7 a)),

$$\vartheta' : \Gamma''' \longrightarrow \Gamma_1^{\mathbb{K}}, \quad z \longmapsto z^2.$$

By the commutativity of the index maps (Axiom 1.2.8) the diagrams

$$\begin{array}{ccccc}
K_i(\mathscr{C}_0(\mathbb{M}'', F)) & \xrightarrow{K_i(\varphi)} & K_i(\mathscr{C}_0(\mathbb{K}', F)) & \xrightarrow{K_i(\psi)} & K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) \\
\downarrow{=} & & \downarrow{K_i(\phi)} & & \downarrow{K_i(\phi')} \\
K_i(\mathscr{C}_0(\mathbb{M}'', F)) & \xrightarrow[K_i(\bar{\varphi}'')]{} & K_i(\mathscr{C}_0(\mathbb{M}'''), F)) & \xrightarrow[K_i(\bar{\psi}'')]{} & K_i(\mathscr{C}(\Gamma''', F))
\end{array}$$

$$\begin{array}{ccc}
K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) & \xrightarrow{\delta_i} & K_{i+1}(\mathscr{C}_0(\mathbb{M}'', F)) \\
\downarrow{K_i(\phi')} & & \downarrow{=} \\
K_i(\mathscr{C}(\Gamma''', F)) & \xrightarrow[\bar{\delta}_i'']{} & K_{i+1}(\mathscr{C}_0(\mathbb{M}'', F))
\end{array}$$

are commutative. By Proposition 3.3.4 c),

$$K_i(\phi') : K_i\left(\mathscr{C}\left(\Gamma^{\mathbb{K}}, F\right)\right) \longrightarrow K_i\left(\mathscr{C}\left(\Gamma''', F\right)\right), \quad (a, b) \longmapsto (a, 2b).$$

By b),

$$\delta_i : K_i\left(\mathscr{C}\left(\Gamma_1^{\mathbb{K}}, F\right)\right) \longrightarrow K_{i+1}\left(\mathscr{C}_0\left(\mathbb{K}'', F\right)\right), \quad (a, b) \longmapsto (a, .2b).$$

d) By the six-term axiom (Axiom 1.2.7), ψ is K-null. The other assertions follow from c) and the six-term axiom (Axiom 1.2.7). ■

3.5 Pasting Locally Compact Spaces

PROPOSITION 3.5.1 *Let Ω_1, Ω_2 be locally compact spaces, Γ_1 and Γ_2 closed sets of Ω_1 and Ω_2, respectively, $\vartheta : \Gamma_1 \longrightarrow \Gamma_2$ a homeomorphism, Ω' the topological sum of $\Omega_1 \setminus \Gamma_1$ and $\Omega_2 \setminus \Gamma_2$, Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega \in \Gamma_1$, Γ the closed set of Ω corresponding to the identified Γ_1 and Γ_2 (so $\Omega \setminus \Gamma = \Omega'$), $\varphi : \mathscr{C}_0(\Omega \setminus \Gamma, F) \longrightarrow \mathscr{C}_0(\Omega, F)$ the inclusion map,*

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Gamma, F), \quad x \longmapsto x|\Gamma,$$

and δ_i the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Gamma, F) \longrightarrow 0 .$$

Let $J := \{1, 2\}$ and for every $j \in J$ let

$$\varphi_j : \mathscr{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0(\Omega_j, F),$$

$$\varphi'_j : \mathscr{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0\left(\Omega', F\right),$$

$$\varphi''_j : \mathscr{C}_0(\Omega_j \setminus \Gamma_j, F) \longrightarrow \mathscr{C}_0(\Omega, F)$$

be the inclusion maps,

$$\psi_j : \mathscr{C}_0(\Omega_j, F) \longrightarrow \mathscr{C}_0(\Gamma_j, F), \quad x \longmapsto x|\Gamma_j,$$

$$\psi'_j : \mathscr{C}_0\left(\Omega', F\right) \longrightarrow \mathscr{C}_0(\Omega_j \setminus \Gamma_j, F), \quad x \longmapsto x|(\Omega_j \setminus \Gamma_j),$$

and $\delta_{j,i}$ the index maps associated to the exact sequence in $\mathfrak{M}_E$

$$0 \longrightarrow \mathscr{C}_0(\Omega_j \setminus \Gamma_j, F) \xrightarrow{\varphi_j} \mathscr{C}_0(\Omega_j, F) \xrightarrow{\psi_j} \mathscr{C}_0(\Gamma_j, F) \longrightarrow 0 .$$

a) $\delta_{j,i} = K_{i+1}(\psi'_j) \circ \delta_i$ for every $j \in J$ and

$$\delta_i = K_{i+1}(\varphi'_1) \circ \delta_{1,i} + K_{i+1}(\varphi'_2) \circ \delta_{2,i} .$$

b) Assume $\mathscr{C}_0(\Omega_1, F)$ K-null.

b_1) $\delta_{1,i} : K_i(\mathscr{C}_0(\Gamma_1, F)) \longrightarrow K_{i11}(\mathscr{C}_0(\Omega_1 \setminus \Gamma_1, F))$ is a group isomorphism.

b_2) δ_i is injective.

b_3) ψ is K-null.

b_4) $K_i(\varphi_2'') : K_i(\mathscr{C}_0(\Omega_2 \setminus \Gamma_2, F)) \longrightarrow K_i(\mathscr{C}_0(\Omega, F))$ is a group isomorphism.

b_5) If we put

$$\Phi_i := K_i(\varphi_2') \circ K_i(\varphi_2'')^{-1} : K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right)$$

then the map

$$K_{i+1}(\mathscr{C}_0(\Gamma, F)) \times K_i(\mathscr{C}_0(\Omega, F)) \longrightarrow K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right),$$

$$(a, b) \longmapsto \delta_{i+1}a + \Phi_i b$$

is a group isomorphism.

b_6) If also $\mathscr{C}_0(\Omega_2, F)$ is K-null then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F)),$$

$$K_i\left(\mathscr{C}_0\left(\Omega', F\right)\right) \approx K_{i+1}(\mathscr{C}_0(\Gamma, F))^2 .$$

a) follows from Proposition 2.3.7 a), since ψ_j'' of this Proposition is the identity map in the present case.

b_1) follows from the Topological six-term sequence (Proposition 2.1.8 a)).

b_2) Let $a \in K_i(\mathscr{C}_0(\Gamma, F))$ such that $\delta_i a = 0$. By a), $\delta_{1,i}a = K_{i+1}(\psi_1')\delta_i a = 0$ and by b_1), $a = 0$.

b_3) follows from b_2) and the six-term axiom (Axiom 1.2.7).

b_4) and b_5) follow from b_3) and Proposition 2.3.7 c_1), c_2).

b_6) follows from b_1), b_4), and the Product Theorem (Proposition 2.3.1 a)). ■

COROLLARY 3.5.2 *Let Γ be a locally compact space, $(\Omega_j)_{j\in J}$ a nonempty finite family of locally compact spaces such that $\mathscr{C}_0(\Omega_j, F)$ is K-null for every $j \in J$, and for every $j \in J$ let Γ_j be a closed set of Ω_j and $\vartheta_j : \Gamma \longrightarrow \Gamma_j$ a homeomorphism. Let Ω' the topological sum of the family $(\Omega_j \setminus \Gamma_j)_{j\in J}$, and Ω the locally compact space obtained* *from the topological sum of the family* $(\Omega_j)_{j\in J}$ *by identifying for every* $\omega\in\Gamma$ *all the points* $\vartheta_j(\omega)$ $(j\in J)$. *Then*

$$K_i(\mathscr{C}_0(\Omega,F))\approx K_{i+1}(\mathscr{C}_0(\Gamma,F))^{Card\,J-1},$$

$$K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right)\approx K_i(\mathscr{C}_0(\Omega,F))\times K_{i+1}(\mathscr{C}_0(\Gamma,F))\approx K_{i+1}(\mathscr{C}_0(\Gamma,F))^{Card\,J}.$$

We prove the Corollary by induction with respect to $Card\,J$. For $Card\,J\in\{1,2\}$ the assertion follows from Proposition 3.5.1 b_1),b_5),b_6). Let $k\in J$, assume the assertion holds for $J':=J\setminus\{\mathrm{k}\}$, and denote by Ω'' the topological sum of the family $(\Omega_j\setminus\Gamma_j)_{j\in J'}$. By Proposition 3.5.1 b_4),b_5) and the induction hypothesis,

$$K_i(\mathscr{C}_0(\Omega,F))\approx K_i\left(\mathscr{C}_0\left(\Omega'',F\right)\right)\approx K_{i+1}(\mathscr{C}_0(\Gamma,F))^{Card\,J-1},$$

$$K_i\left(\mathscr{C}_0\left(\Omega',F\right)\right)\approx K_{i+1}(\mathscr{C}_0(\Gamma,F))\times K_i\left(\mathscr{C}_0\left(\Omega'',F\right)\right)\approx K_{i+1}(\mathscr{C}_0(\Gamma,F))^{J}.$$ ■

COROLLARY 3.5.3 *Let* $m,n\in\mathbb{N}$,

$$\Gamma_+:=\{\,\alpha\in\mathbb{B}_n\mid\|\alpha\|=1,\,\alpha_n>0\,\},\,\Gamma_-:=\{\,\alpha\in\mathbb{B}_n\mid\|\alpha\|=1,\,\alpha_n\le 0\,\},$$

and Ω *the locally compact space obtained from the topological sum of the family* $(\mathbb{B}_n\setminus\Gamma_-)_{j\in\mathbb{N}_m}$ *by identifying all the* Γ_+. *Then*

$$K_i(\mathscr{C}_0(\Omega,F))\approx K_{i+n}(F)^{m-1}.$$

By Proposition 2.4.1, $\mathscr{C}_0(\mathbb{B}_n\setminus\Gamma_-,F)$ is null-homotopic and so K-null. For $n>1$, Γ_+ is homeomorphic to $\mathbb{R}^{n-1}$ so by Theorem 3.2.2 a),

$$K_i(\mathscr{C}_0(\Gamma_+,F))\approx K_{i+n-1}(F)$$

and this relation obviously holds also for $n=1$. Then by Corollary 3.5.2,

$$K_i(\mathscr{C}_0(\Omega,F))\approx K_{i+n}(F)^{m-1}.$$ ■

Remark. The above result can be deduced also from Example 2.4.9 by using Proposition 1.5.11 d).

COROLLARY 3.5.4 *Let Ω', Ω'' be locally compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the locally compact space obtained from the topological sum of Ω' and Ω'' by identifying ω' and ω''. If $\mathscr{C}_0(\Omega'', F)$ is K-null then*

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\omega'\}, F\right)\right) .$$

The assertion follows from Proposition 3.5.1 b_4). ■

PROPOSITION 3.5.5 *Let Ω', Ω'' be compact spaces, $\omega' \in \Omega'$, $\omega'' \in \Omega''$, and Ω the compact space obtained by identifying the points ω' and ω'' in the topological sum of Ω' and Ω''. Then*

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \Omega', F\right)\right) \times K_i\left(\mathscr{C}\left(\Omega', F\right)\right) .$$

Let $\varphi : \mathscr{C}_0(\Omega \setminus \Omega', F) \longrightarrow \mathscr{C}(\Omega, F)$ be the inclusion map and

$$\psi : \mathscr{C}(\Omega, F) \longrightarrow \mathscr{C}\left(\Omega', F\right), \quad x \longmapsto x|\Omega' .$$

We put for every $x \in \mathscr{C}(\Omega', F)$,

$$\lambda x : \Omega \longrightarrow F, \quad \omega \longmapsto \begin{cases} x(\omega) & \text{if} \quad \omega \in \Omega' \\ x(\omega_0) & \text{if} \quad \omega \in \Omega'' \end{cases} ,$$

where $\omega_0 \in \Omega$ denotes the point corresponding to the identified points ω' and ω''. Then

$$0 \longrightarrow \mathscr{C}_0\left(\Omega \setminus \Omega', F\right) \xrightarrow{\varphi} \mathscr{C}(\Omega, F) \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \mathscr{C}\left(\Omega', F\right) \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$ and the assertion follows from the split exact axiom (Axiom 1.2.3). ■

PROPOSITION 3.5.6 *Let $(\Omega_j)_{j \in \mathbb{N}_n}$ be a family of compact spaces and for every $j \in \mathbb{N}_n$ let ω_j, ω'_j be distinct points of Ω_j. If Ω denotes the compact space obtained from the topological sum of the family $(\Omega_j)_{j \in \mathbb{N}_n}$ by identifying ω'_j with ω_{j+1} for all $j \in \mathbb{N}_{n-1}$ then*

$$K_i(\mathscr{C}(\Omega, F)) \approx K_i(F) \times \prod_{j=1}^{n} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_j\}, F\right)) .$$

If $(k_j)_{j \in \mathbb{N}_n}$ is a family in $\mathbb{N}$, $\Omega_j = \mathbf{S}_{k_j}$ for every $j \in \mathbb{N}_n$, and

$$p := Card\left\{\, j \in \mathbb{N}_n \mid k_j \text{ is even} \right\}, \qquad q := Card\left\{\, j \in \mathbb{N}_n \mid k_j \text{ is odd} \right\}$$

then

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{p+1} \times K_{i+1}(F)^q .$$

We put $\bar{\Omega}_n := \Omega$ and prove the assertion by induction with respect to $n \in \mathbb{N}$. For $n = 1$ the assertion follows from the Alexandroff K-theorem (Theorem 2.2.1 a)). Assume the assertion hods for an $n \in \mathbb{N}$. By Proposition 3.5.5 and the induction hypothesis,

$$K_i\left(\mathscr{C}\left(\bar{\Omega}_{n+1},F\right)\right) \approx K_i\left(\mathscr{C}_0\left(\bar{\Omega}_{n+1} \setminus \bar{\Omega}_n,F\right)\right) \times K_i\left(\mathscr{C}\left(\bar{\Omega}_n,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_{n+1} \setminus \{\omega_{\mathrm{n+1}}\},F)) \times K_i\left(\mathscr{C}\left(\bar{\Omega}_n,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_{n+1} \setminus \{\omega_{\mathrm{n+1}}\},F)) \times K_i(F) \times \prod_{j=1}^{n} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_{\mathrm{j}}\},F\right)) \approx$$

$$\approx K_i(F) \times \prod_{j=1}^{n+1} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_{\mathrm{j}}\},F\right)),$$

which finishes the inductive proof. The last assertion follows now from Theorem 3.2.2 a), since $\mathbf{S}_{k_j} \setminus \{\omega_{\mathrm{j}}\}$ is homeomorphic to $\mathbb{R}^{k_j}$. ■

PROPOSITION 3.5.7 *Let Ω_1, Ω_2 be locally compact spaces such that the $E-C*-$algebra $\mathscr{C}_0(\Omega_2,F)$ is K-null, Γ a compact set of Ω_1, and $\vartheta : \Gamma \longrightarrow \Omega_2$ a continuous map. We denote by Ω the locally compact space obtained from the topological sum of Ω_1 and Ω_2 by identifying the points ω and $\vartheta(\omega)$ for all $\omega \in \Gamma$.*

a) If

$$\varphi : \mathscr{C}_0(\Omega_1 \setminus \Gamma,F) \longrightarrow \mathscr{C}_0(\Omega,F)$$

denotes the inclusion map then

$$K_i(\varphi) : K_i(\mathscr{C}_0(\Omega_1 \setminus \Gamma,F)) \longrightarrow K_i(\mathscr{C}_0(\Omega,F))$$

is a group isomorphism. If in addition $\Omega \in \Upsilon$ or $\Omega_1 \setminus \Gamma \in \Upsilon$ then

$$\Omega, \Omega_1 \setminus \Gamma \in \Upsilon,\ p(\Omega) = p(\Omega_1 \setminus \Gamma),\ q(\Omega) = q(\Omega_1 \setminus \Gamma),\ \Omega_\Upsilon = (\Omega_1 \setminus \Gamma)_\Upsilon .$$

b) If Ω^* denotes the Alexandroff compactification of Ω then

$$K_i(\mathscr{C}(\Omega^*,F)) \approx K_i(F) \times K_i(\mathscr{C}_0(\Omega_1 \setminus \Gamma,F)) .$$

a) If we put

$$\psi : \mathscr{C}_0(\Omega, F) \longrightarrow \mathscr{C}_0(\Omega_2, F), \quad x \longmapsto x|\Omega_2$$

then

$$0 \longrightarrow \mathscr{C}_0(\Omega_1 \setminus \Gamma, F) \xrightarrow{\varphi} \mathscr{C}_0(\Omega, F) \xrightarrow{\psi} \mathscr{C}_0(\Omega_2, F) \longrightarrow 0$$

is an exact sequence in $\mathfrak{M}_E$. Since $\mathscr{C}_0(\Omega_2, F)$ is K-null, the assertion follows from the Topological six-term sequence (Proposition 2.1.8 c)).

b) follows from a) and Alexandroff's K-theorem (Theorem 2.2.1 a)). ■

COROLLARY 3.5.8 *Let $(\Omega_j)_{j\in J}$ be a finite family of locally compact spaces, $\omega_j \in \Omega_j$ for all $j \in J$, and Ω the locally compact space obtained from the topological sum of the family $(\Omega_j)_{j\in J}$ by identifying the points ω_j for all $j \in J$.*

a) If there is a $j_0 \in J$ such that $\mathscr{C}_0\left(\Omega_{j_0}, F\right)$ is K-null then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx \prod_{j\in J\setminus\{j_0\}} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_j\}, F\right)).$$

b) If $\Omega_j := [0,1[$ for all $j \in J$ and $n := Card\, J$ then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_{i+1}(F)^{n-1}.$$

c) Let $j_0 \in J$ and $\Omega_{j_0} := [0,1[$. If $(k_j)_{j\in J\setminus\{j_0\}}$ is a family in $\mathbb{N}$,

$$p := Card\left\{\, j \in J \setminus \{j_0\} \mid k_j \text{ is even} \right\},$$

$$q := Card\left\{\, j \in J \setminus \{j_0\} \mid k_j \text{ is odd} \right\},$$

and $\Omega_j := \mathbf{S}_{k_j}$ for every $j \in J \setminus j_0$ then

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i(F)^p \times K_{i+1}(F)^q.$$

a) Let Ω' be the locally compact space obtained from the topological sum of the family $(\Omega_j)_{J\setminus\{j_0\}}$ by identifying the points ω_j for all $j \in J \setminus \{j_0\}$ and let $\bar{\omega}$ denote the point obtained by this identification. If we replace in Proposition 3.5.7 Ω_1 by Ω', Ω_2 by Ω_{j_0}, Γ by $\bar{\omega}$, and take $\vartheta(\bar{\omega}) := \omega_{j_0}$ then we get

$$K_i(\mathscr{C}_0(\Omega, F)) \approx K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\bar{\omega}\}, F\right)\right).$$

$\Omega' \setminus \{\bar{\omega}\}$ is the topological sum of the family $(\Omega_j \setminus \{\omega_{\mathrm{j}}\})_{j \in J \setminus \{\mathrm{j}_0\}}$ so by the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega' \setminus \{\bar{\omega}\}, F\right)\right) \approx \prod_{j \in J \setminus \{\mathrm{j}_0\}} K_i(\mathscr{C}_0\left(\Omega_j \setminus \{\omega_{\mathrm{j}}\}, F\right)) .$$

b) follows immediately from a) since $\mathscr{C}_0\left([0,1[, F\right)$ is K-null and

$$K_i(\mathscr{C}_0\left([0,1[\setminus\{\omega\}, F\right)) \approx K_{i+1}(F)$$

for all $\omega \in [0,1[$.

c) For $j \in J \setminus \{\mathrm{j}_0\}$, $\mathbf{S}_{k_j} \setminus \{\omega_{\mathrm{j}}\}$ is homeomorphic to $\mathbb{R}^{k_j}$ and so by Theorem 3.2.2 a), $K_i\left(\mathscr{C}_0\left(\mathbf{S}_{k_j} \setminus \{\omega_{\mathrm{j}}\}, F\right)\right) \approx K_{i+k_j}(F)$. Since $\mathscr{C}_0\left([0,1[, F\right)$ is K-null, we get from a),

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(F)^p \times K_{i+1}(F)^q . \qquad \blacksquare$$

COROLLARY 3.5.9 *Let J_1, J_2, J_3 be pairwise disjoint finite sets and let Ω be the locally compact space (the* **graph***) obtained from the topological sum of* $[0,1] \times J_1$, $[0,1[\times J_2$, *and* $]0,1[\times J_3$ *by identifying some of the points of the set*

$$\{\ (0,j) \mid j \in J_1 \cup J_2\ \} \cup \{\ (1,j) \mid j \in J_1\ \} .$$

If s denotes the number of compact connected components of Ω and r_0 and r_1 denote the number of vertices and chords of the graph Ω, respectively, then

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(F)^s \times K_{i+1}(F)^{s+r_1-r_0} .$$

By the Product Theorem (Proposition 2.3.1 a)), we may assume Ω connected.

Assume first there is a $j \in J_3$ such that Ω contains $]0,1[\times\{\mathrm{j}\}$. Since Ω is connected, $\Omega =]0,1[\times\{\mathrm{j}\}$. Thus Ω is homeomorphic to $\mathbb{R}$, $r_1 - r_0 = 1$, and the assertion follows from Theorem 3.2.2 a).

Assume now there is a $j \in J_2$ such that Ω contains $[0,1[\times\{\mathrm{j}\}$. By Proposition 3.5.7 a),

$$K_i(\mathscr{C}_0\left(\Omega, F\right)) \approx K_i(\mathscr{C}_0\left(\Omega \setminus ([0,1[\times\{\mathrm{j}\}), F\right)) .$$

Ω and $\Omega \setminus ([0,1[\times\{\mathrm{j}\})$ have the same $r_1 - r_0$, so we may replace Ω by $\Omega \setminus ([0,1[\times\{\mathrm{j}\})$. Repeating the operation, we obtain finally a locally compact space, which is the topological sum of a finite family $(]0,1[)_{j\in J}$, and in this case the assertion follows from the Product Theorem (Proposition 2.3.1 a)) and Theorem 3.2.2 a).

Finally assume Ω compact. Then there is a $j \in J_1$ such that Ω contains $[0,1]\times\{\mathrm{j}\}$. By the above and by Alexandroff's K-theorem

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_i(\mathscr{C}_0(\Omega\setminus\{(1,\mathrm{j})\},F))\ .$$

If s', r'_0, r'_1 denote the corresponding numbers associated to $\Omega\setminus\{(1,\mathrm{j})\}$ then $s'=0$, $r'_0 = r_0 - 1$, and $r'_1 = r_1$. All the connected components of $\Omega\setminus\{(1,\mathrm{j})\}$ satisfy the condition of the above paragraphs, so

$$K_i(\mathscr{C}_0(\Omega\setminus\{(1,\mathrm{j})\},F)) \approx K_{i+1}(F)^{r'_1-r'_0} \approx K_{i+1}(F)^{1+r_1-r_0}\,,$$

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+1}(F)^{1+r_1-r_0}\ .$$

■

COROLLARY 3.5.10 *If Ω is a compact graph contained in $\mathbb{B}_n$ then*

$$K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega,F)) \approx K_i(F)^{s-r_0+r_1}\times K_{i+1}(F)^{s-1}\,,$$

where s denotes the number of connected components of Ω and r_0 and r_1 the munber of vertices and chords of Ω, respectively.

Let ω be a vertex of Ω. By Corollary 3.5.9 and Corollary 2.4.4 a),

$$K_i(\mathscr{C}_0(\Omega\setminus\{\omega\},F)) \approx K_i(F)^{s-1}\times K_{i+1}(F)^{s-r_0+r_1}$$

and by Theorem 3.1.2 b),

$$K_i(\mathscr{C}_0(\mathbb{B}_n\setminus\Omega,F)) \approx K_{i+1}(\mathscr{C}_0(\Omega\setminus\{\omega\},F)) \approx K_i(F)^{s-r_0+r_1}\times K_{i+1}(F)^{s-1}\ .$$

■

EXAMPLE 3.5.11 *Let $n\in\mathbb{N}$, Γ a closed set of $\mathbf{S}_n$, $\emptyset\neq\Gamma\neq\mathbf{S}_n$, $\omega\in\Gamma$, Γ' the compact space obtained from $\Gamma\times[0,1]$ by identifying the points of $\Gamma\times 0$, and Ω the compact space obtained from the topological sum of $\mathbf{S}_n$ and Γ' by identifying the points of $\Gamma\subset\mathbf{S}_n$ with the points of $\Gamma\times\{1\}\subset\Gamma'$.*

a) $K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+n}(F)\times K_{i+1}(\mathscr{C}_0(\Gamma\setminus\{\omega\},F))$.

b) If Γ is finite then

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)\times K_{i+n}\left(F\right)\times K_{i+1}\left(F\right)^{Card\,\Gamma-1} .$$

c) If Γ is a graph then

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)^{1+s+r_1-r_0}\times K_{i+n}\left(F\right)\times K_{i+1}\left(F\right)^{s-1} ,$$

where s denotes the number of connected components of Ω and r_0 and r_1 denote the number of vertices and chords of the graph Γ, respectively,

a) By Theorem 3.2.2 e_1),

$$K_i\left(\mathscr{C}_0\left(\mathbf{S}_n\setminus\Gamma,F\right)\right)\approx K_{i+n}\left(F\right)\times K_{i+1}\left(\mathscr{C}_0\left(\Gamma\setminus\{\omega\},F\right)\right) .$$

By Proposition 2.4.1, $\mathscr{C}_0\left(\Gamma'\setminus\{0\},F\right)$ is K-null, where 0 is the point obtained from the identification of the points of $\Gamma\times\{0\}$. By Proposition 3.5.7 a),

$$K_i\left(\mathscr{C}_0\left(\Omega\setminus\{0\},F\right)\right)\approx K_i\left(\mathscr{C}_0\left(\mathbf{S}_n\setminus\Gamma,F\right)\right) ,$$

so by Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)\times K_{i+n}\left(F\right)\times K_{i+1}\left(\mathscr{C}_0\left(\Gamma\setminus\{\omega\},F\right)\right) .$$

b) follows from a) and the Product Theorem (Proposition 2.3.1 a)).

c) By Corollary 3.5.9 and Alexandroff's K-theorem (Theorem 2.2.1 a)),

$$K_i\left(\mathscr{C}\left(\Gamma,F\right)\right)\approx K_i\left(F\right)^s\times K_{i+1}\left(F\right)^{s+r_1-r_0} ,$$

$$K_i\left(\mathscr{C}_0\left(\Gamma\setminus\{\omega\},F\right)\right)\approx K_i\left(F\right)^{s-1}\times K_{i+1}\left(F\right)^{s+r_1-r_0} ,$$

so by a),

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)^{1+s+r_1-r_0}\times K_{i+n}\left(F\right)\times K_{i+1}\left(F\right)^{s-1} .$$

■

PROPOSITION 3.5.12 *Let $(p_j)_{j\in J}$ be a finite family in $\mathbb{N}$, ($J\neq\emptyset$), and for every $j\in J$ put $\Omega_j:=\mathbf{S}_{p_j}$. Let Ω' be the topological sum of the family $(\Omega_j)_{j\in J}$, $(\Gamma_k)_{k\in K}$ a finite family of pairwise disjoint nonempty finite subsets of Ω', $\Gamma:=\bigcup_{k\in K}\Gamma_k$, and Ω the compact space obtained from Ω' by identifying for every $k\in K$ the points of Γ_k. If Ω is connected then*

$$K_i\left(\mathscr{C}\left(\Omega,F\right)\right)\approx K_i\left(F\right)\times K_{i+1}\left(F\right)^{Card\,\Gamma-Card\,J-Card\,K+1}\times\prod_{j\in J}K_{i+p_j}\left(F\right) .$$

If $K = \emptyset$, since Ω is connected, J is a one-point set and the assertion holds by Theorem 3.2.2 b). Thus we may assume $K = \mathbb{N}_n$ for some $n \in \mathbb{N}$. Take $k_1 \in K$ and put $J_1 := \left\{ j \in J \mid \Omega_j \cap \Gamma_{k_1} \neq \emptyset \right\}$. We define recursively an injective family $(k_m)_{m\in\mathbb{N}_n}$ in K and an increasing family $(J_m)_{m\in\mathbb{N}_n}$ of subsets of J in the following way. Let $m \in \mathbb{N}_n$, $m > 1$, and assume the families were defined up to $m-1$. Since Ω is connected there is a $k_m \in K \setminus \left\{ k_q \mid q \in \mathbb{N}_{m-1} \right\}$ such that $\Gamma_{k_m} \cap J_{m-1} \neq \emptyset$. We put

$$J_m := \left\{ j \in J \mid \Omega_j \cap \left(\bigcup_{q=1}^{m} \Gamma_{k_q} \right) \neq \emptyset \right\} .$$

It is easy to prove by induction with respect to $m \in \mathbb{N}_n$ that

$$Card \left(\bigcup_{q=1}^{m} \Gamma_{k_q} \right) - Card\, J_m - m + 1 \geq 0$$

for every $m \in \mathbb{N}_n$. In particular,

$$Card\, \Gamma - Card\, J - Card\, K + 1 \geq 0 .$$

For every $j \in J$, by Proposition 2.4.11 and Theorem 3.2.2 a),

$$K_i\left(\mathscr{C}_0\left(\Omega_j \setminus \Gamma, F\right)\right) \approx K_{i+1}\left(F\right)^{Card\,(\Gamma\cap\Omega_j)-1} \times K_{i+p_j}\left(F\right)$$

so that by the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega' \setminus \Gamma, F\right)\right) \approx K_{i+1}\left(F\right)^{Card\,\Gamma - Card\,J} \times \prod_{j\in J} K_{i+p_j}\left(F\right) .$$

For every $k \in K$ let ω_k be the point of Ω corresponding to the unified points of Γ_k and put $\Delta := \{ \omega_k \mid k \in K \}$. Then by Proposition 2.4.11,

$$K_i\left(\mathscr{C}_0\left(\Omega \setminus \Delta, F\right)\right) \approx K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega_{k_0}\}, F\right)\right) \times K_{i+1}\left(F\right)^{Card\,K-1} ,$$

where $k_0 \in K$. By the above and by Alexandroff's K-theorem, since $\Omega \setminus \Delta = \Omega' \setminus \Gamma$,

$$K_i\left(\mathscr{C}\left(\Omega, F\right)\right) \times K_{i+1}\left(F\right)^{Card\,K-1} \approx$$

$$\approx K_i\left(F\right) \times K_i\left(\mathscr{C}_0\left(\Omega \setminus \{\omega_{k_0}\}, F\right)\right) \times K_{i+1}\left(F\right)^{Card\,K-1} \approx$$

$$\approx K_i\left(F\right) \times K_i\left(\mathscr{C}_0\left(\Omega \setminus \Delta, F\right)\right) \approx K_i\left(F\right) \times K_i\left(\mathscr{C}_0\left(\Omega' \setminus \Gamma, F\right)\right) \approx$$

$$\approx K_i\left(F\right) \times K_{i+1}\left(F\right)^{Card\,\Gamma - Card\,J - Card\,K + 1} \times K_{i+1}\left(F\right)^{Card\,K-1} \times \prod_{j\in J} K_{i+p_j}\left(F\right) ,$$

$$K_i\left(\mathscr{C}\left(\Omega, F\right)\right) \approx K_i\left(F\right) \times K_{i+1}\left(F\right)^{Card\,\Gamma - Card\,J - Card\,K + 1} \times \prod_{j\in J} K_{i+p_j}\left(F\right) . \qquad \blacksquare$$

COROLLARY 3.5.13 *Let $(p_j)_{j\in\mathbb{N}_n}$ be a family in $\mathbb{N}$ and for every $j\in\mathbb{N}_n$ put $\Omega_j := \mathbb{S}_{p_j}$. For every $j\in\mathbb{N}_n$ let Γ_j and Γ'_j be disjoint nonempty finite subsets of Ω_j such that $k_j := Card\,\Gamma'_j = Card\,\Gamma_{j+1}$ for every $j\in\mathbb{N}_{n-1}$. We denote by Ω the compact space obtained from the topological sum of the family $(\Omega_j)_{j\in\mathbb{N}_n}$ by identifying in a bijective way Γ'_j with Γ_{j+1} for all $j\in\mathbb{N}_{n-1}$. Then*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)\times K_{i+1}(F)^{\sum_{j=1}^{n-1}(k_j-1)} \times \prod_{j=1}^{n} K_{i+p_j}(F)\,. \qquad \blacksquare$$

PROPOSITION 3.5.14 *Let Ω_1, Ω_2 be locally compact spaces and for every $j\in\{1,2\}$ let Γ_j be a compact set of Ω_j and $\vartheta_j : \mathbb{B}_n \longrightarrow \Gamma_j$ a homeomorphism such that $\Delta_j := \vartheta_j(\mathbb{B}_n\setminus\mathbb{S}_{n-1})$ is an open set of Ω_j. We denote by Ω the locally compact space obtained from the topological sum of $\Omega_1\setminus\Delta_1$ and $\Omega_2\setminus\Delta_2$ by identifying the points $\vartheta_1(\omega)$ and $\vartheta_2(\omega)$ for all $\omega\in\mathbb{S}_{n-1}$. Then for every $\omega\in\mathbb{S}_{n-1}$,*

$$K_i(\mathscr{C}_0(\Omega\setminus\{\vartheta_1(\omega)\},F)) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\times K_{i+n-1}(F)\,.$$

We use the notation of the topological triple (Proposition 2.1.11), which we mark with a prime in order to distinguish them from the present notation. We put $\Omega'_2 := \Omega\setminus\{\vartheta_1(\omega)\}$ and take as Ω'_3 the topological sum of $\Omega_1\setminus\Gamma_1$ and $\Omega_2\setminus\Gamma_2$ and as Ω'_1 the locally compact space obtained from Ω by completing first $\vartheta_1(\mathbb{S}_{n-1})$ to $\vartheta_1(\mathbb{B}_n)$ and deleting then ω. By the Product Theorem (Proposition 2.3.1 a)),

$$K_i\left(\mathscr{C}_0\left(\Omega'_3,F\right)\right) \approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\,.$$

Since $\Omega'_2\setminus\Omega'_3$ is homeomorphic to $\mathbb{S}_{n-1}\setminus\{\omega\}$, we get by Theorem 3.2.2 e_1),

$$K_i\left(\mathscr{C}_0\left(\Omega'_2\setminus\Omega'_3,F\right)\right) \approx K_{i+n-1}(F)\,.$$

Thus by the topological triple (Proposition 2.1.11 b_3)) (and Theorem 3.1.2 b)),

$$K_i(\mathscr{C}_0(\Omega\setminus\{\vartheta_1(\omega)\},F)) \approx K_i\left(\mathscr{C}_0\left(\Omega'_2,F\right)\right) \approx$$

$$\approx K_i\left(\mathscr{C}_0\left(\Omega'_3,F\right)\right)\times K_i\left(\mathscr{C}_0\left(\Omega'_2\setminus\Omega'_3,F\right)\right) \approx$$

$$\approx K_i(\mathscr{C}_0(\Omega_1\setminus\Gamma_1,F))\times K_i(\mathscr{C}_0(\Omega_2\setminus\Gamma_2,F))\times K_{i+n-1}(F)\,. \qquad \blacksquare$$

COROLLARY 3.5.15 *If S_g is an orientable compact connected surface of genus $g \in \mathbb{N}$ and Γ is a nonempty finite subset of S_g then*

$$K_i(\mathscr{C}(S_g,F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g-1} ,$$

$$K_i(\mathscr{C}(S_g \setminus \Gamma,F)) \approx K_i(F)^{g} \times K_{i+1}(F)^{3g-2+Card\,\Gamma} .$$

Assume first Γ is a one-point set $\{\omega\}$. We prove the second assertion in this case by induction with respect to $g \in \mathbb{N}$. By Proposition 3.2.15 b), the assertion holds for $g = 1$. Assume now the assertion holds for $g \in \mathbb{N}$. Let Δ_1 be a closed disc of S_1, Δ_g a closed disc of S_g, $\omega \in \Delta_1$, and $\omega \in \Delta_g$. $S_{g+1} \setminus \{\omega\}$ can be obtained from the topological sum of $S_1 \setminus \Delta_1$, $S_g \setminus \Delta_2$, and $\mathbf{S}_1 \setminus \{\omega\}$ by pasting $\mathbf{S}_1 \setminus \{\omega\}$ in the the boundaries of $\Delta_1 \setminus \{\omega\}$ and $\Delta_g \setminus \{\omega\}$. By the induction hypothesis, since $S_g \setminus \Delta_g$ is homeomorphic to $S_g \setminus \{\omega\}$,

$$K_i(\mathscr{C}_0(S_g \setminus \Delta_g,F)) \approx K_i(F)^{g} \times K_{i+1}(F)^{3g-1} .$$

By Proposition 3.5.14,

$$K_i(\mathscr{C}_0(S_{g+1} \setminus \{\omega\},F)) \approx K_i(F)^{g+1} \times K_{i+1}(F)^{3g+2} ,$$

which finishes the inductive proof.

The first assertion follows now from Alexandroff's K-theorem (Proposition 2.2.1 a)) and the second one from Proposition 2.4.11. ■

The following Example shows a way to generalize Corollary 3.5.15.

EXAMPLE 3.5.16 *Let Ω be the compact space obtained from the topological sum of $\mathbf{S}_1 \times \mathbf{S}_2 \setminus \Delta$, $\mathbf{S}_1 \times \mathbf{S}_1 \times \mathbf{S}_1 \setminus \Delta'$, and $\mathbf{S}_2$, where Δ and Δ' denote balls homeomorphic to $\mathbb{B}_3$ by pasting $\mathbf{S}_2$ in the boundaries of Δ and Δ'. Then for every nonempty finite subset Γ of Ω,*

$$K_i(\mathscr{C}(\Omega,F)) \approx K_i(F)^{5} \times K_{i+1}(F)^{6} ,$$

$$K_i(\mathscr{C}_0(\Omega \setminus \Gamma,F)) \approx K_i(F)^{4} \times K_{i+1}(F)^{5+Card\,\Gamma} .$$ ■

Remark. Let

$$0 \longrightarrow F_1 \stackrel{\varphi}{\longrightarrow} F_2 \stackrel{\psi}{\longrightarrow} F_3 \longrightarrow 0 ,$$

$$0 \longrightarrow G_1 \xrightarrow{\varphi'} G_2 \xrightarrow{\psi'} G_3 \longrightarrow 0$$

be exact sequences in $\mathfrak{M}_E$ and $\lambda : F_3 \longrightarrow G_3$ and isomorphism in $\mathfrak{M}_E$. Then

$$H := \left\{ (x,y) \in F_2 \times G_2 \mid \psi' y = \lambda \psi x \right\}$$

is a C*-subalgebra of $F_2 \times G_2$ containing the ideal $F_1 \times G_1$ of $F_2 \times G_2$. H corresponds to the operation of pasting F_2 and G_2 in $\mathfrak{M}_E$.

Chapter 4

Some Supplementary Results

Throughout this chapter F denotes an E-C*-algebra.

4.1 Full E-C*-algebras

DEFINITION 4.1.1 *A* **full E-C*-algebra** *is a unital C*-algebra F for which E is a* canonical *unital C*-subalgebra such that $\alpha x = x\alpha$ for all $(\alpha, x) \in E \times F$. Every full E-C*-algebra is canonically an E-C*-algebra, the exterior multiplication being the restriction of the interior multiplication. We denote by $\mathfrak{C}_E$ the category of full E-C*-algebras for which the morphisms are the unital E-linear C*-homomorphisms. In particular $\mathfrak{C}_{\mathbb{C}}$ is the category of all unital C*-algebras with unital C*-homomorphisms. A* **full E-C*-subalgebra of** *F is a C*-subalgebra of F containing E. An isomorphism of full E-C*-algebras is also called* **E-C*-isomorphism**.

If $\prod_{j\in J} F_j$ is a finite family of full E-C-algebras, $J \neq \emptyset$, then $\prod_{j\in J} F_j$ is a full E-C*-algebra, the canonical embedding $E \to \prod_{j\in J} F_j$ being given by*

$$E \longrightarrow \prod_{j\in J} F_j, \quad \alpha \longmapsto (\alpha)_{j\in J} .$$

If F is a full E-C*-algebra and G a unital C*-algebra then the map

$$E \longrightarrow F \otimes G, \quad \alpha \longmapsto \alpha \otimes 1_G$$

is an injective C*-homomorphism. In particular, the E-C*-algebra $F \otimes G$ has a canonical structure of a full E-C*-algebra.

PROPOSITION 4.1.2 *Let F be an E-C*-algebra. We denote by $\check{F}$ the vector space $E \times F$ endowed with the bilinear map*

$$(E\times F)\times(E\times F) \longrightarrow E\times F, \quad ((\alpha,x),(\beta,y)) \longmapsto (\alpha\beta, \alpha y + \beta x + xy)$$

and with the involution

$$E\times F \longrightarrow E\times F, \quad (\alpha,x) \longmapsto (\alpha^*, x^*) .$$

a) $\check{F}$ is an involutive unital algebra with $(1_E, 0)$ as unit and $\{\,(\alpha,0) \mid \alpha \in E\}$ is a unital involutive subalgebra of $\check{F}$ isomorphic to E.

b) *If E and F are C*-subalgebras of a C*-algebra G then the map*

$$\varphi : \check{F} \longrightarrow E \times G, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective involutive algebra homomorphism with closed image

$$\{ (\alpha, y) \in E \times G \mid \alpha - y \in F \} .$$

In particular $\varphi(\check{F})$ is a C-subalgebra of $E \times G$ and there is a norm on $\check{F}$ with respect to which $\check{F}$ is a C*-algebra.*

c) *There is a unique C*-norm on $\check{F}$ making it a C*-algebra. Moreover $\check{F}$ is a full E-C*-algebra and F may be identified with the closed ideal*

$$\{ (0, x) \mid x \in F \}$$

of $\check{F}$. We shall always consider $\check{F}$ endowed with the structure of a full E-C-algebra.*

d) *If F is a full E-C*-algebra then the map*

$$\check{F} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an isomorphism of E-C-algebras with inverse*

$$E \times F \longrightarrow \check{F}, \quad (\alpha, x) \longmapsto (\alpha, x - \alpha) .$$

e) *If $E = \mathbb{C}$ then $\check{F}$ is the unitization $\tilde{F}$ of F.*

a) is easy to verify.

b) Only the assertion that the image of φ is closed needs a proof. Let $(\alpha, x) \in \overline{\varphi(\check{F})}$. There are sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ in E and F, respectively, such that

$$\lim_{n \to \infty} (\alpha_n, \alpha_n + x_n) = (\alpha, x) .$$

It follows

$$\alpha = \lim_{n \to \infty} \alpha_n \in E , \quad x - \alpha = \lim_{n \to \infty} x_n \in F , \quad (\alpha, x) = \varphi(\alpha, x - \alpha) \in \varphi(\check{F}) .$$

Thus $\varphi(\check{F})$ is closed.

c) Let Ω be the spectrum of E and $\tilde{F}$ the unitization of F. Then E and F are C*-subalgebras of the C*-algebra $\mathscr{C}\left(\Omega,\tilde{F}\right)$ and the assertion follows from b).

d) follows from c) and b).

e) is obvious. ■

EXAMPLE 4.1.3 *Let F be a commutative E-C*-algebra.*

a) *$\check{F}$ is commutative. We denote by Ω_E, Ω_F, and $\Omega_{\check{F}}$ the spectra of E, F, and $\check{F}$, respectively.*

b) *Ω_F is homeomorphic to an open set Ω' of $\Omega_{\check{F}}$ such that $F \approx \mathscr{C}_0(\Omega',\mathbb{C})$.*

c) *There is a unique surjective continuous map $\vartheta : \Omega_{\check{F}} \longrightarrow \Omega_E$ such that if we put*

$$\phi : E \approx \mathscr{C}(\Omega_E,\mathbb{C}) \longrightarrow \check{F} \approx \mathscr{C}(\Omega_{\check{F}},\mathbb{C}) , \quad \alpha \longmapsto \alpha\circ\vartheta$$

then ϕ is an injective continuous C-homomorphism (so we may identify E with $\phi(E)$).*

d) *The restriction of ϑ to $\Omega_{\check{F}}\setminus\Omega'$ is a homeomorphism.*

e) *If F is unital then $\Omega_{\check{F}}$ is homeomorphic to the topological sum of Ω_E and Ω_F.*

a) is easy to see.

b) follows from the fact that F may be identified with a closed ideal of $\check{F}$ (Proposition 4.1.2 c)).

c) is proved in [1] Proposition 4.1.2.15.

d) Let $\omega \in \Omega_E$ and put

$$\omega' : \check{F} \longrightarrow \mathbb{C}, \quad (\alpha,x) \longmapsto \alpha(\omega) .$$

Then $\omega' \in \Omega_{\check{F}}\setminus\Omega'$ and $\vartheta(\omega') = \omega$, so $\vartheta|(\Omega_{\check{F}}\setminus\Omega')$ is surjective.

Let $\omega_1,\omega_2 \in \Omega_{\check{F}}\setminus\Omega'$, $\omega_1 \neq \omega_2$. There is an $(\alpha,x) \in \check{F}$ with

$$\langle(\alpha,x),\omega_1\rangle \neq \langle(\alpha,x),\omega_2\rangle .$$

Since $\langle(\alpha,x),\omega_j\rangle = \langle\alpha,\omega_j\rangle$ for every $j\in\{1,2\}$, $\vartheta|(\Omega_{\check{F}}\setminus\Omega')$ is injective.

e) follows from d) since in this case Ω' is clopen. ■

Remark. The above d) may be seen as a kind of generalization of Alexandroff's compactification.

DEFINITION 4.1.4 *We put for every E-C*-algebra F*

$$\iota^F : F \longrightarrow \check{F}\,, \quad x \longmapsto (0,x)\,,$$

$$\pi^F : \check{F} \longrightarrow E\,, \quad (\alpha,x) \longmapsto \alpha\,,$$

$$\lambda^F : E \longrightarrow \check{F}\,, \quad \alpha \longmapsto (\alpha,0)\,,$$

$$\sigma^F := \lambda^F \circ \pi^F\,.$$

If $E=\mathbb{C}$ then

$$\check{F} = \tilde{F}\,, \qquad \iota_F = \iota^F\,, \qquad \pi_F = \pi^F\,, \qquad \lambda_F = \lambda^F\,.$$

All these maps are E-linear C*-homomorphisms,

$$\pi^F \circ \iota^F = 0, \qquad \pi^F \circ \lambda^F = id_E, \qquad \pi^F \circ \sigma^F = \pi^F\,,$$

ι^F and λ^F are injective, π^F, λ^F, and σ^F are unital, and

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\overleftarrow{\lambda^F}}{\overset{\pi^F}{\rightarrow}} E \longrightarrow 0$$

is a split exact sequence in $\mathfrak{M}_E$.

PROPOSITION 4.1.5

a) If $F \overset{\varphi}{\rightarrow} F'$ is a morphism in $\mathfrak{M}_E$ then the map

$$\check{\varphi} : \check{F} \longrightarrow \check{F}'\,, \quad (\alpha,x) \longmapsto (\alpha,\varphi x)$$

is an involutive unital algebra homomorphism, injective or surjective if φ is so. If $F=F'$ and if φ is the identity map then $\check{\varphi}$ is also the identity map.

b) *Let F_1, F_2, F_3 be E-C*-algebras and let $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ be E-linear C*-homomorphisms. Then* $\overset{\vee}{\overbrace{\psi \circ \varphi}} = \check{\psi} \circ \check{\varphi}$. ■

Remark. If $E = \mathbb{C}$ then $\check{\varphi} = \tilde{\varphi}$.

EXAMPLE 4.1.6 *Let F be a full E-C*-algebra and F' a closed ideal of F.*

a) *F' endowed with the exterior multiplication*

$$E \times F' \longrightarrow F', \quad (\alpha, x) \longmapsto \alpha x$$

is an E-C-algebra.*

b) *The map*

$$\check{F'} \longrightarrow E \times F, \quad (\alpha, x) \longmapsto (\alpha, \alpha + x)$$

is an injective E-linear C-homomorphism with image*

$$\left\{ (\alpha, x) \in E \times F \mid \alpha - x \in F' \right\} .$$

c) *$\mathfrak{C}_E$ is a full subcategory of $\mathfrak{M}_E$.* ■

PROPOSITION 4.1.7 *Let F be a full E-C*-algebra and J a finite set.*

a) *$F^J = F \otimes l^2(J)$ endowed with the maps*

$$F \times F^J \longrightarrow F^J, \quad (x, \xi) \longmapsto (x\xi_j)_{j \in J},$$

$$F^J \times F \longrightarrow F^J, \quad (\xi, x) \longmapsto (\xi_j x)_{j \in J},$$

$$F^J \times F^J \longrightarrow F, \quad (\xi, \eta) \longmapsto \sum_{j \in J} \eta_j^* \xi_j$$

is a unital Hilbert F-module ([1] Proposition 5.6.4.2 c)).

b) *Let $\mathscr{L}(F^J)$ be the Banach space of operators on F^J. The set $\mathscr{L}_F(F^J)$ of adjointable operators on F^J is a Banach subspace of $\mathscr{L}(F^J)$. $\mathscr{L}_F(F^J)$ endowed with the restriction of the norm of $\mathscr{L}(F^J)$ it is a full E-C*-algebra* ([1] Theorem 5.6.1.11 d), [1] Proposition 5.6.1.8 g),h)). ■

PROPOSITION 4.1.8 *For every E-C*-algebra F the sequence*

$$0 \longrightarrow K_i(F) \xrightarrow{K_i(\iota^F)} K_i(\check{F}) \underset{\xleftarrow{K_i(\lambda^F)}}{\xrightarrow{K_i(\pi^F)}} K_i(E) \longrightarrow 0$$

is split exact and the map

$$K_i(F) \times K_i(E) \longrightarrow K_i(\check{F}), \quad (a,b) \longmapsto K_i(\iota^F)\, a + K_i(\lambda^F)\, b$$

is a group isomorphism.

Since the sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \xrightarrow{\iota^F} \check{F} \underset{\xleftarrow{\lambda^F}}{\xrightarrow{\pi^F}} E \longrightarrow 0$$

is split exact the assertion follows from the split exact axiom (Axiom 1.2.3). ■

COROLLARY 4.1.9 *Let G be a C*-algebra.*

a) The sequence in $\mathfrak{M}_E$

$$0 \longrightarrow F \otimes G \xrightarrow{\iota^F \otimes id_G} \check{F} \otimes G \underset{\xleftarrow{\lambda^F \otimes id_G}}{\xrightarrow{\pi^F \otimes id_G}} E \otimes G \longrightarrow 0$$

is split exact.

b) The sequence

$$0 \longrightarrow K_i(F \otimes G) \xrightarrow{K_i(\iota^F \otimes id_G)} K_i(\check{F} \otimes G) \underset{\xleftarrow{K_i(\lambda^F \otimes id_G)}}{\xrightarrow{K_i(\pi^F \otimes id_G)}} K_i(E \otimes G) \longrightarrow 0$$

is split exact and the map

$$K_i(E \otimes G) \times K_i(F \otimes G) \longrightarrow K_i(\check{F} \otimes G),$$

$$(a,b) \longmapsto K_i(\lambda^F \otimes id_G)\, a + K_i(\iota^F \otimes id_G)\, b$$

is a group isomorphism.

c) Let $F \xrightarrow{\varphi} F'$ *be a morphism in* $\mathfrak{M}_E$ *and* $G \xrightarrow{\psi} G'$ *a morphism in* $\mathfrak{M}_{\mathbf{C}}$. *If we identify the isomorphic groups of b) then*

$$K_i(\check{\varphi} \otimes \psi) : K_i(\check{F} \otimes G) \longrightarrow K_i(\check{F}' \otimes G'),$$

$$(a,b) \longmapsto (K_i(id_E \otimes \psi)\, a, K_i(\varphi \otimes \psi)\, b)$$

is a group isomorphism.

a) follows from Proposition 1.4.8 a).

b) follows from a) and the split exact axiom (Axiom 1.2.3).

c) follows from b) and the commutativity of the following diagram:

$$\begin{array}{ccccc} F\otimes G & \xrightarrow{\iota^F\otimes id_G} & \check{F}\otimes G & \xleftarrow{\lambda^F\otimes id_G} & E\otimes G \\ {\scriptstyle \varphi\otimes\psi}\downarrow & & {\scriptstyle \check{\varphi}\otimes\psi}\downarrow & & \downarrow{\scriptstyle id_E\otimes\psi} \\ F'\otimes G' & \xrightarrow[\iota^{F'}\otimes id_{G'}]{} & \check{F}'\otimes G' & \xleftarrow[\lambda^{F'}\otimes id_{G'}]{} & E\otimes G' \end{array} .$$

■

COROLLARY 4.1.10 *Let $F \overset{\phi_1}{\to} F'$ and $F \overset{\phi_2}{\to} F'$ be morphisms in $\mathfrak{M}_E$. If F is K-null then $K_i\left(\check{\phi}_1\right) = K_i\left(\check{\phi}_2\right)$.*

By Proposition 4.1.8, the map

$$K_i(F)\times K_i(E) \longrightarrow K_i\left(\check{F}\right), \quad (a,b)\longmapsto K_i\left(\iota^F\right)a + K_i\left(\lambda^F\right)b$$

is a group isomorphism. Since F is K-null, $K_i\left(\lambda^F\right)$ is a group isomorphism. We get from $\check{\phi}_1\circ\lambda^F = \check{\phi}_2\circ\lambda^F$,

$$K_i\left(\check{\phi}_1\right)\circ K_i\left(\lambda^F\right) = K_i\left(\check{\phi}_2\right)\circ K_i\left(\lambda^F\right), \qquad K_i\left(\check{\phi}_1\right) = K_i\left(\check{\phi}_2\right) .$$

■

4.2 Continuity and Stability

AXIOM 4.2.1 (Continuity axiom) *If $\{(F_j)_{j\in J}, (\varphi_{j,k})_{j,k\in I}\}$ is an inductive system in $\mathfrak{M}_E$ such that $\varphi_{j,k}$ are injective for all $j,k\in J$, $k<j$, and if $\{F, (\varphi_j)_{j\in J}\}$ denotes its inductive limit in $\mathfrak{M}_E$ then $\{K_i(F), (K_i(\varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F_j))_{j\in J}, (K_i(\varphi_{j,k}))_{j,k\in J}\}$.*

PROPOSITION 4.2.2 *If Ω is a totally disconnected compact space then*

$$K_i(\mathscr{C}(\Omega,F)) \approx \left\{ a\in K_i(F)^\Omega \;\middle|\; a(\Omega) \text{ is finite} \right\} .$$

Let Ξ be the set of clopen partitions of Ω ordered by fineness and for every $\Theta := (\Omega_j)_{j \in J} \in \Xi$ and $x \in F^\Theta$ put

$$\tilde{x} : \Omega \longrightarrow F\,, \quad \omega \longmapsto x(j) \quad \text{for} \quad \omega \in \Omega_j\,.$$

Then the map

$$F^\Theta \longrightarrow \mathscr{C}(\Omega, F), \quad x \longmapsto \tilde{x}$$

is an injective E-C*-homomorphism for every $\Theta \in \Xi$ and $\mathscr{C}(\Omega, F)$ is isomorphic to the corresponding inductive limit in $\mathfrak{M}_E$ of $(F^\Theta)_{\Theta \in \Xi}$. By Lemma 2.1.4 c), $K_i(F^\Theta) \approx K_i(F)^\Theta$ for every $\Theta \in \Xi$ and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

PROPOSITION 4.2.3 *Let ξ be an ordinal number, $(\Omega_\eta)_{\eta<\xi}$ a family of path connected, non-compact, locally compact spaces, and $\omega_\eta \in \Omega_\eta$ for every $\eta < \xi$. We denote by Ω^ξ the locally compact space obtained by endowing the disjoint union of the family of sets $(\Omega_\eta)_{\eta<\xi}$ with the topology for which a subset U of Ω^ξ is open if it has the following properties:*

1) *$\Omega_\eta \cap U$ is open for every $\eta < \xi$.*
2) *If $\omega_\eta \in U$ for some $\eta < \xi$ and if there is a $\zeta < \eta$ with $\eta = \zeta + 1$ then $\Omega_\zeta \setminus U$ is compact.*
3) *If $\omega_\eta \in U$ for some limit ordinal number $\eta < \xi$ then there is a $\zeta < \eta$ such that $\bigcup_{\zeta<\zeta'<\eta} \Omega_{\zeta'} \subset U$.*

If $K_i(\mathscr{C}_0(\Omega_\eta, F)) = 0$ for all $\eta < \xi$ then $K_i\left(\mathscr{C}_0\left(\Omega^\xi, F\right)\right) = 0$.

The assertion is trivial for $\xi = 0$. We prove the general case by transfinite induction. If $\xi = \eta + 1$ for some $\eta < \xi$ for which the assertion holds then by Corollary 3.5.4, the assertion holds also for ξ. If ξ is a limit ordinal number and the assertion holds for every $\eta < \xi$ then by the continuity axiom (Axiom 4.2.1) the assertion holds also for ξ since $\mathscr{C}_0\left(\Omega^\xi, F\right)$ is the inductive limit of the inductive system $\{\, \mathscr{C}_0(\Omega^\eta, F) \mid \eta < \xi \,\}$. ■

Remark. If $\Omega_\eta = [0,1[$ for every $\eta < \xi$ then Ω^ξ is "one-dimensional".

LEMMA 4.2.4 *Let $\{(F_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_E$, $\{F, (\varphi_j)_{j\in J}\}$ its inductive limit in $\mathfrak{M}_E$, G an E-C*-algebra, and for every $j \in J$ an injective morphism $\psi_j : F_j \longrightarrow G$ in $\mathfrak{M}_E$ such that $\psi_j = \psi_k \circ \varphi_{k,j}$ for all $j,k \in J$, $j < k$. Then the morphism $\psi : F \longrightarrow G$ in $\mathfrak{M}_E$ such that $\psi_j = \psi \circ \varphi_j$ for all $j,k \in J$, $j < k$,* ([5] Theorem L.2.1) *is injective.*

For $j \in J$ and $x \in F_j$,

$$\left\|\varphi_j x\right\| \leq \|x\| = \left\|\psi_j x\right\| = \left\|\psi\varphi_j x\right\| \leq \left\|\varphi_j x\right\|,$$

so ψ preserves the norms on $\varphi_j(F_j)$. Since $\bigcup_{j\in J} \varphi_j(F_j)$ is dense in F, ψ preserves the norms, i.e. it is injective. ■

PROPOSITION 4.2.5 *Let $\{(G_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{j,k}$ are injective for all $j,k \in J$, $k < j$, and let $\{G, (\varphi_j)_{j\in J}\}$ be its inductive limit in $\mathfrak{M}_{\mathbf{C}}$. If $\{F', (\psi_j)_{j\in J}\}$ denotes the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(F \otimes G_j)_{j\in J}, (id_F \otimes \varphi_{j,k})_{j,k\in J}\}$ in $\mathfrak{M}_E$ and $\psi : F' \longrightarrow F \otimes G$ denotes the morphism in $\mathfrak{M}_E$ such that $\psi \circ \psi_j = id_F \otimes \varphi_j$ for all $j \in J$* ([5] Theorem L.2.1) *then ψ is an isomorphism.*

By [5] Corollary T.5.19, $id_F \otimes \varphi_j$ are injective for all $j \in J$. By Lemma 4.2.4, ψ is injective. Since

$$F \otimes \left(\bigcup_{j\in J} G_j\right) \subset Im\,\psi,$$

ψ is surjective and so it is an isomorphism. ■

COROLLARY 4.2.6 *If $\{(G_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{j,k}$ are injective for all $j,k \in J$, $k < j$, and if $\{G, (\varphi_j)_{j\in J}\}$ is its inductive limit in $\mathfrak{M}_{\mathbf{C}}$ then $\{K_i(F \otimes G), (K_i(id_F \otimes \varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F \otimes G_j))_{j\in J}, (K_i(id_F \otimes \varphi_{j,k}))_{j,k\in J}\}$. In particular if G_j is Υ-null for every $j \in J$ then G is also Υ-null.*

By [5] Corollary T.5.19, $id_F \otimes \varphi_{j,k}$ are injective for all $j,k \in J$, $k < j$. By Proposition 4.2.5, $\{F \otimes G, (id_F \otimes \varphi_j)_{j\in J}\}$ may be identified with the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(F \otimes G_j)_{j\in J}, (id_F \otimes \varphi_{j,k})_{j,k\in J}\}$ in $\mathfrak{M}_E$ and the assertion follows from the continuity axiom (Axiom 4.2.1). ■

COROLLARY 4.2.7 *Let $(G_j)_{j\in J}$ be an infinite family in Υ_1, $\mathfrak{J}$ the set of nonempty finite subsets of J ordered by inclusion, and for all $K,L\in\mathfrak{J}$, $K\subset L$, put $G_K := \bigotimes_{j\in K} G_j$ and*

$$\varphi(L,K) : G_K \longrightarrow G_L\,,\quad \bigotimes_{j\in K} x_j \longmapsto \bigotimes_{j\in L} y_j\,,$$

where

$$y_j := \begin{cases} x_j & \text{if} \quad j\in K \\ 1_{G_j} & \text{if} \quad j\in L\setminus K \end{cases}.$$

Then $\{(G_K)_{K\in\mathfrak{J}}, (\varphi(L,K))_{K,L\in\mathfrak{J}}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ and its limit belongs to Υ_1.

We denote by $\{G, (\varphi(K))_{K\in\mathfrak{J}}\}$ the above inductive limit. By Proposition 1.6.5, $G_K\in\Upsilon_1$ for all $K\in\mathfrak{J}$ so by Corollary 4.2.6, $p(G)=1$, $q(G)=0$. Let $F\xrightarrow{\phi}F'$ be a morphism in $\mathfrak{M}_E$ and let $K\in\mathfrak{J}$. Then the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\phi_{G_K,F}} & F\otimes G_K & \xrightarrow{id_F\otimes\varphi(K)} & F\otimes G \\ \phi\downarrow & & \downarrow\phi\otimes id_{G_K} & & \downarrow\phi\otimes id_G \\ F' & \xrightarrow[\phi_{G_K,F'}]{} & F'\otimes G_K & \xrightarrow[id_{F'}\otimes\varphi(K)]{} & F'\otimes G \end{array}$$

is commutative. Since

$$\phi_{G,F} = (id_F\otimes\varphi(K))\circ\phi_{G_K,F}\,,\qquad \phi_{G,F'} = (id_{F'}\otimes\varphi(K))\circ\phi_{G_K,F'}\,,$$

the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\phi_{G,F}} & F\otimes G \\ \phi\downarrow & & \downarrow\phi\otimes id_G \\ F' & \xrightarrow[\phi_{G,F'}]{} & F'\otimes G \end{array}\qquad \begin{array}{ccc} K_i(F) & \xrightarrow{K_i(\phi_{G,F})} & K_i(F\otimes G) \\ K_i(\phi)\downarrow & & \downarrow K_i(\phi\otimes id_G) \\ K_i(F') & \xrightarrow[K_i(\phi_{G,F'})]{} & K_i(F'\otimes G) \end{array}$$

are commutative and so $G\in\Upsilon_1$. ■

COROLLARY 4.2.8 *Let $\{(G_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J}\}$ be an inductive system in $\mathfrak{M}_{\mathbf{C}}$ such that $\varphi_{k,j}$ are injective for all $j,k\in J$, $j<k$, and let $\{G, (\varphi_j)_{j\in J}\}$ be its inductive limit. We*

assume that for all $j,k \in J$, $j < k$,

$$G_j, G_k \in \Upsilon, \qquad \Phi_{i,G_k,F} = K_i\left(id_F \otimes \varphi_{k,j}\right) \circ \Phi_{i,G_j,F} \, .$$

Then

$$G \in \Upsilon, \qquad \Phi_{i,G,F} = K_i\left(id_F \otimes \varphi_j\right) \circ \Phi_{i,G_j,F}$$

for all $j \in J$.

By Corollary 4.2.6, $\{K_i(F \otimes G), (K_i(id_F \otimes \varphi_j))_{j\in J}\}$ is the inductive limit of the inductive system $\{(K_i(F \otimes G_j))_{j\in J}, (K_i(id_F \otimes \varphi_{j,k}))_{j,k\in J}\}$. By the hypothesis of the Corollary,

$$K_i\left(id_F \otimes \varphi_{k,j}\right) : K_i\left(F \otimes G_j\right) \longrightarrow K_i\left(F \otimes G_k\right)$$

is a group isomorphism for all $j,k \in J$, $j < k$, so

$$K_i\left(id_F \otimes \varphi_j\right) : K_i\left(F \otimes G_j\right) \longrightarrow K_i\left(F \otimes G\right)$$

is also a group isomorphism for all $j \in J$. Let $F \xrightarrow{\phi} F'$ be a morphism in $\mathfrak{M}_E$. The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc}
K_i(F)^{p(G_j)} \times K_{i+1}(F)^{q(G_j)} & \xrightarrow{K_i(\phi)^{p(G_j)} \times K_{i+1}(\phi)^{q(G_j)}} & A \\
\Phi_{i,G_j,F}\big\downarrow & & \Phi_{i,G_j,F'}\big\downarrow \\
K_i\left(F \otimes G_j\right) & \xrightarrow{K_i\left(\phi \otimes G_j\right)} & K_i\left(F' \otimes G_j\right) \\
K_i\left(id_F \otimes \varphi_j\right)\big\downarrow & & K_i\left(id_{F'} \otimes \varphi_j\right)\big\downarrow \\
K_i(F \otimes G) & \xrightarrow[K_i(\phi \otimes id_G)]{} & K_i\left(F' \otimes G\right)
\end{array}$$

where $A := K_i(F')^{p(G_j)} \times K_{i+1}(F')^{q(G_j)}$. ■

DEFINITION 4.2.9 *We denote for every family $(\mathscr{G}_j)_{j\in J}$ of additive groups by $\sum_{j\in J} \mathscr{G}_j$ its direct sum i.e.*

$$\sum_{j\in J} \mathscr{G}_j := \left\{ a \in \prod_{j\in J} \mathscr{G}_j \;\middle|\; \left\{ j \in J \mid a_j \neq 0 \right\} \text{ is finite} \right\} .$$

PROPOSITION 4.2.10 *If* $(F_j)_{j\in J}$ *is a family of E-C*-algebras and* F *is its C*-direct sum* ([1] Example 4.1.1.6) *then*

$$K_i(F) \approx \sum_{j\in J} K_i(F_j) \ .$$

In particular, the C-direct sum of a family of K-null E-C*-algebras is K-null.*

If J is finite then the assertion follows from Proposition 1.3.3. The general case follows now from the continuity (Axiom 4.2.1). ■

COROLLARY 4.2.11 *If* $(\Omega_j)_{j\in J}$ *is a family of locally compact spaces and* Ω *is its topological sum then*

$$K_i(\mathscr{C}_0(\Omega,F)) \approx \sum_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F)) \approx$$

$$\approx \left\{ a \in \prod_{j\in J} K_i(\mathscr{C}_0(\Omega_j,F)) \;\middle|\; \{\, j\in J \mid a_j \neq 0\,\} \text{ is finite} \right\} .$$

By Proposition 4.2.5, $\mathscr{C}_0(\Omega,F)$ is the direct sum of the family $(\mathscr{C}_0(\Omega_j,F))_{j\in J}$ and the assertion follows from Proposition 4.2.10. ■

PROPOSITION 4.2.12 *If* ξ *is an ordinal number endowed with its usual topology then* $K_i(\mathscr{C}_0(\xi,F)) \approx \sum\limits_{\eta<\xi} K_i(F)$.

We prove the assertion by transfinite induction. If ξ is not a limit ordinal number then the assertion follows from Corollary 2.3.4 a). Assume ξ is a limit ordinal number and for all $\eta<\zeta<\xi$ let $\varphi_{\zeta,\eta}:\mathscr{C}_0(\eta,F)\longrightarrow\mathscr{C}_0(\zeta,F)$ be the inclusion map. By Proposition 4.2.5, $\mathscr{C}_0(\xi,F)$ may be identified with the inductive limit in $\mathfrak{M}_E$ of the inductive system $\{(\mathscr{C}_0(\eta,\mathrm{F}))_{\eta<\xi},\ (\varphi_{\zeta,\eta})_{\eta<\zeta<\xi}\}$ in $\mathfrak{M}_E$. Thus the assertion follows from the continuity axiom (Axiom 4.2.1) and the induction hypothesis. ■

DEFINITION 4.2.13 *We denote for every* $n\in\mathbb{N}$ *by* $M(n)$ *the C*-algebra of* $n\times n$*-matrices with entries in* $\mathbb{C}$.

AXIOM 4.2.14 (Stability axiom) *There is an* $h \in \mathbb{N}$, $h \neq 1$, *such that*

$$M(h) \in \Upsilon\,, \qquad p(M(h)) = 1\,, \qquad q(M(h)) = 0\,,$$

$$\Phi_{i,M(h),F} = K_i\,(id_F \otimes \varphi) \circ \Phi_{i,\mathbb{C},F}\,,$$

where

$$\varphi : \mathbb{C} \longrightarrow M(h)\,, \quad \alpha \longmapsto \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

PROPOSITION 4.2.15 *We put for all* $j,k \in \mathbb{N}^*$, $j < k$,

$$\varphi_{k,j} : M(h^j) \longrightarrow M(h^k)\,, \quad x \longmapsto \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

a) For all $j \in \mathbb{N}$,

$$M(h^j) \in \Upsilon\,, \qquad p(M(h^j)) = 1\,, \qquad q(M(h^j)) = 0\,,$$

$$\Phi_{i,M(h^j),F} = K_i\left(id_F \otimes \varphi_{j,0}\right) \circ \Phi_{i,\mathbb{C},F}\,.$$

b) For all $j,k \in \mathbb{N}^*$, $j < k$,

$$\Phi_{i,M(h^k),F} = K_i\left(id_F \otimes \varphi_{k,j}\right) \circ \Phi_{i,M(h^j),F}$$

and $K_i\left(id_F \otimes \varphi_{k,j}\right)$ *is a group isomorphism.*

a) We prove the assertion by induction with respect to $j \in \mathbb{N}$. For $j = 1$ the assertion is exactly the Stability axiom (Axiom 4.2.14). Let $j > 1$ and assume the assertion holds for $j - 1$. With the notation of Proposition 1.5.4 b),

$$\left(id_{F \otimes M(h)} \otimes \varphi_{(j-1),0}\right) \circ \phi_{\mathbb{C},F \otimes M(h)} \circ \left(id_F \otimes \varphi_{1,0}\right) = id_F \otimes \varphi_{j,0}\,,$$

so by the above and by the induction hypothesis,

$$K_i\left(id_F \otimes \varphi_{j,0}\right) \circ \Phi_{i,\mathbb{C},F} =$$

$$= K_i\left(id_{F\otimes M(h)}\otimes\varphi_{(j-1),0}\right)\circ\Phi_{i,\mathbb{C},F\otimes M(h)}\circ K_i\left(id_F\otimes\varphi_{1,0}\right)\circ\Phi_{i,\mathbb{C},F} =$$

$$= \Phi_{i,M(h^{j-1}),F\otimes M(h)}\circ\Phi_{i,M(h),F}\ .$$

Thus

$$K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F} : K_i(F)\longrightarrow K_i\left(F\otimes M(h^j)\right)$$

is a group isomorphism. Let $F\overset{\phi}{\to}F'$ be a morphism in $\mathfrak{M}_E$. Since the diagram

$$\begin{array}{ccccc}
K_i(F) & \xrightarrow{\Phi_{i,\mathbb{C},F}} & K_i(F\otimes M(1)) & \xrightarrow{K_i\left(id_F\otimes\varphi_{j,0}\right)} & K_i\left(F\otimes M(h^j)\right)\\
{\scriptstyle K_i(\phi)}\downarrow & & \downarrow{\scriptstyle K_i\left(\phi\otimes id_{M(1)}\right)} & & \downarrow{\scriptstyle K_i\left(\phi\otimes id_{M(h^j)}\right)}\\
K_i(F') & \xrightarrow[\Phi_{i,\mathbb{C},F'}]{} & K_i(F'\otimes M(1)) & \xrightarrow[K_i\left(id_{F'}\otimes\varphi(j,0)\right)]{} & K_i\left(F'\otimes M(h^j)\right)
\end{array}$$

is commutative, we may take

$$\Phi_{i,M(h^j),F} = K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F}\ .$$

b) By a),

$$K_i\left(id_F\otimes\varphi_{k,j}\right)\circ\Phi_{i,M(h^j),F} = K_i\left(id_F\otimes\varphi_{k,j}\right)\circ K_i\left(id_F\otimes\varphi_{j,0}\right)\circ\Phi_{i,\mathbb{C},F} =$$

$$= K_i\left(id_F\otimes\varphi_{k,0}\right) = \Phi_{i,M(h^k),F}\ . \qquad \blacksquare$$

THEOREM 4.2.16 *Let H be an infinite-dimensional Hilbert space and $\mathcal{K}(H)$ the C*-algebra of compact operators on H. Then*

$$\mathcal{K}(H)\in\Upsilon,\qquad p(\mathcal{K}(H))=1,\qquad q(\mathcal{K}(H))=0,$$

$$\Phi_{i,\mathcal{K}(H),F} = K_i\left(id_F\otimes\varphi\right)\circ\Phi_{i,\mathbb{C},F}\ ,$$

where $\varphi:\mathbb{C}\longrightarrow\mathcal{K}(H)$ is an inclusion map.

Let Ξ be the set of subspaces of H of dimension h^j for some $j\in\mathbb{N}^*$ ordered by inclusion and for every $K\in\Xi$ let π_K be the orthogonal projection of H on K and $G_K := \pi_K\mathcal{K}(H)\pi_K$. We denote for all $K,L\in\Xi$, $K\subset L$, by

$$\varphi_{L,K} : G_K\longrightarrow G_L\,,\qquad\qquad \varphi_K : G_K\longrightarrow\mathcal{K}(H)$$

the inclusion maps. Then $\{(G_K)_{K\in\Xi}\,,\ (\varphi_{L,K})_{L,K\in\Xi}\}$ is an inductive system in $\mathfrak{M}_{\mathbf{C}}$ and $\{\mathscr{K}(H)\,,(\varphi_K)_{K\in\Xi}\}$ is its inductive limit. By Proposition 4.2.15, for $K,L\in\Xi$, $K\subset L$,

$$G_K, G_L \in \Upsilon\,, \qquad p(G_K)=p(G_L)=1\,, \qquad q(G_K)=q(G_L)=0\,,$$

$$\Phi_{i,G_L,F} = K_i\,(id_F\otimes\varphi_{L,K})\circ\Phi_{i,G_K,F}\;,$$

and $K_i\,(id_F\otimes\varphi_{L,K})$ is a group isomorphism. By Corollary 4.2.8, for $K\in\Xi$,

$$\mathscr{K}(H)\in\Upsilon\,, \qquad \Phi_{i,\mathscr{K}(H),F} = K_i\,(id_F\otimes\varphi_K)\circ\Phi_{i,G_K,F}\;,$$

so $p(\mathscr{K}(H))=1\,,\ q(\mathscr{K}(H))=0$. ■